On Enforcing Liveness in a Class of Partially-Controlled General Free-Choice Petri Nets via Supervision

N. Somnath* and R. S. Sreenivas†
* Industrial and Enterprise Systems Engineering
† *Senior Member, IEEE,* CSL & Industrial and Enterprise Systems Engineering,
University of Illinois at Urbana-Champaign,
Urbana, IL 61820.

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Abstract

A *Petri net* (PN) is said to be a *Free-Choice PN* (FCPN) if every arc from a place to a transition is either the unique output arc from the place, or is the unique input arc to the transition. A PN is said to be an *ordinary* (*general*) PN if the weights associated with any arc is (not necessarily) unity. In this report we identify a class of general FCPN structures, $\mathcal{F}$, which strictly includes the class of ordinary FCPN structures, where the existence of a supervisory policy that enforces liveness in $N(\mathbf{m}^0)$, where $N \in \mathcal{F}$, is decidable.

Index Terms

Petri Nets, Supervisory Control, Discrete Event Systems.

I. Introduction

A *Petri net* (PN) where are all arc weights are unity is an *ordinary* PN, and a PN without restrictions on the arc weights is a *general* PN (cf. section 5.3, [1]). A PN is a *Free-Choice Petri net* (FCPN) if each arc from a place to a transition is either the unique output arc of the place, or is the unique input arc to the transition. Application of FCPNs include the modeling of product-flow in manufacturing environments (cf. chapter 2, [2]) and flow of control in processor networks (cf. section 1.2, [3]).

A PN is *live* if irrespective of the past transition firings, every transition in the PN can fire at some point in the future. A system modeled by a live PN does not experience *livelocks*, which is a desirable feature. A PN model that is not live, can be made live with help of a supervisory policy that prevents the firing of a select group of transitions at each *marking*. This paper is about supervisory policies that enforce liveness in general FCPNs.

We introduce examples that place the results of this paper in a broader context following some notational preliminaries. For an appropriate index set for the set of places in a PN, a marking can be interpreted as a vector whose i-th component is the number of tokens in the i-th place. The marking can also be interpreted as a function that assigns a non-negative number of tokens to the places in a PN. The informal discussion in the remainder of this section uses the function- and vector-interpretation of the marking of a PN interchangeably. We assume some familiarity with PNs, and we refer the reader to references [1], [4] for a detailed exposition of various topics related to PNs. The technical terms that are used in this section should be interpreted colloquially, they are formally defined in section II.

In the graphical representation of PNs, the transitions that are represented by filled (unfilled) rectangles can (cannot) be prevented from firing by the supervisory policy. For instance, transition t_1 in the ordinary

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FCPN structure shown in figure 1(a) is the only transition that can be prevented from firing at any marking. The remaining transitions in this PN can fire uncontrollably if they are enabled. Also, for brevity we only represent the non-unity arc weights along side each arc in graphical representations of general PNs in this paper. That is, if there is an arc without an explicit arc weight in any graphical representation, then the arc weight is unity.

If there are transitions that can never be prevented from firing in a PN, then there is no *positive-test* for the existence, or non-existence, of a liveness enforcing supervisory policy [5]. The existence of a liveness enforcing supervisory policy is decidable if the PN is bounded, or if each transition in an unbounded PN can be prevented from firing when deemed necessary [6]. Additionally, the existence of a supervisory policy that enforces liveness in ordinary FCPNs is decidable [5].

A common feature in the decidable instances, not including the class of bounded PNs, is that the set of initial markings for which there is a liveness enforcing supervisory policy in an instance, is *right-closed*. That is, if there is a supervisory policy that enforces liveness in a PN structure belonging to one of these classes for an initial marking, then there is a liveness enforcing policy for the same PN structure when it is initialized with a larger initial marking. The first result in this paper is about a condition on general FCPNs that guarantees the right-closure of the set of initial markings for which there is a liveness enforcing supervisory policy.

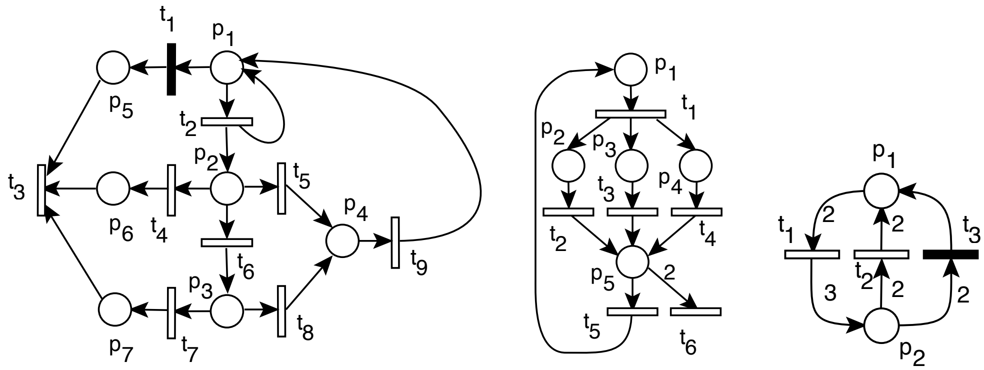
There is a supervisory policy that enforces liveness in the ordinary FCPN structure N_1 shown in figure 1(a) if and only if the initial marking $\mathbf{m}_1^0$ is greater than or equal to one of two *minimal elements* in the set $\{(1\ 0\ 0\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0\ 0\ 0)^T\}$. For any initial marking $\mathbf{m}_1^0$ that satisfies this requirement, the supervisory policy that permits the firing of transition t_1 at a marking if and only if either there are two or more tokens in p_1, or there is at least one token in p_4, enforces liveness in $N(\mathbf{m}_1^0)$.

However, the set of initial markings for which there is a supervisory policy that enforces liveness in general FCPN structures is not necessarily right-closed. Consider the general FCPN structure N_2 shown in figure 1(b), the arc (p_5, t_6) has a weight of two. Since all transitions in N_2 are uncontrollable, there is a (trivial) supervisory policy that enforces liveness in $N_2(\mathbf{m}_2^0)$ if and only if $N_2(\mathbf{m}_2^0)$ is live. Since $N_2(\mathbf{m}_2^0)$ is live if and only if the sum of the tokens assigned to all places at initialization is an odd number, it follows that the set of initial markings for which there is a liveness enforcing supervisory policy for N_2 is not right-closed.

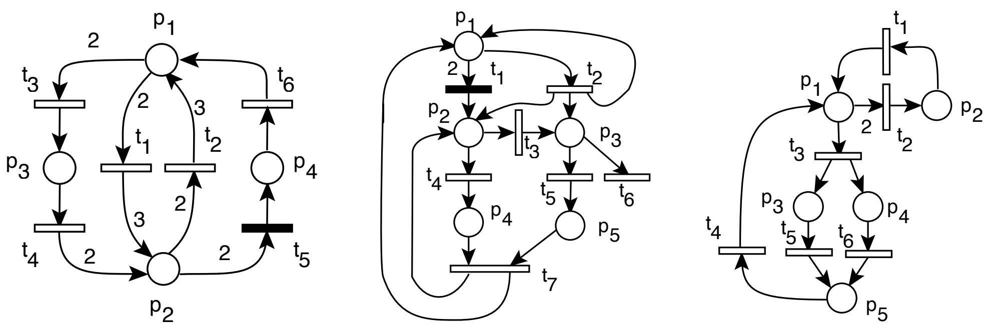
In this paper we identify a class, $\mathcal{F}$, of general FCPN structures, for which the set of initial markings such that there is a liveness enforcing supervisory policy, is right-closed. The class $\mathcal{F}$ strictly includes ordinary FCPN structures. As a consequence of the results in this paper, it follows that the existence of a supervisory policy that enforces liveness in the class $\mathcal{F}$ is decidable. Each member of $\mathcal{F}$ is identified by the following property – if a place has multiple output transitions, at least one of which is uncontrollable, then the weight(s) associated with the arc(s) that originate from the place at hand, to each uncontrollable transition, must be the smallest of all outgoing arc weights from the place. That is, if an output transition of a place in a member of $\mathcal{F}$ is state-enabled at a marking, then every uncontrollable output transition of this place should also be state-enabled at the same marking.

For example, transitions t_2 and t_3 are output transitions of p_2 in the FCPN structure N_3 shown in figure 1(c). Both outgoing arcs from p_2 have a weight of two, and therefore the weight associated with the arc from p_2 to the uncontrollable transition t_2 is the smallest weight of all arcs originating from place p_2, and this FCPN structure belongs to $\mathcal{F}$. It can be shown that the set of initial markings for which there is a supervisory policy that enforces liveness in N_3 is the right-closed set with minimal elements $\{(2\ 0)^T, (0\ 2)^T\}$. The minimally restrictive supervisory policy that enforces liveness prevents the firing of transition t_3 at a marking, if the new marking that would result from its firing is not in the right-closed set with minimal elements $\{(2\ 0)^T, (0\ 2)^T\}$. Since the set of initial markings is *control-invariant*, the marking that would result from the firing of uncontrollable transitions t_1 and t_2 from any marking in this right-closed set is guaranteed to remain in the right-closed set of markings.

The general FCPN structure N_4 shown in figure 1(d) (cf. figure 2c, [7]) also belongs to the class $\mathcal{F}$, as every outgoing arc from place p_1 or p_2 has a weight of two. There is a liveness-enforcing policy



(a) Ordinary FCPN structure N_1 (b) General FCPN structure N_2 (c) General FCPN structure N_3



(d) General FCPN structure N_4 (e) General FCPN structure N_5 (f) General FCPN structure N_6

Fig. 1. (a) There is a supervisory policy that enforces liveness if N_1 is initialized with markings that are greater than or equal to one of the two minimal elements in the set $\{(1\ 0\ 0\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0\ 0\ 0)^T\}$. (b) The set of initial markings for which there is a supervisory policy that enforces liveness in N_2 is not right-closed. This is because N_2 is live for any initial marking such that the sum of the tokens assigned to all places at initialization is an odd number. This general FCPN is not a member of the class $\mathcal{F}$ as t_6 is an uncontrollable transition, and the weight associated with the arc (p_5, t_6) is not the smallest of the weights associated with arcs originating from p_5. (c) The set of initial markings for which there is a supervisory policy that enforces liveness for N_3 is the right-closed set of markings that are greater than or equal to one of the two minimal elements in the set $\{(2\ 0)^T, (0\ 2)^T\}$. This general FCPN is a member of the class $\mathcal{F}$. (d) This general FCPN structure is from figure 2c in reference [7]. The right-closed set of initial markings for which there is a supervisory policy that enforces liveness is defined by the minimal elements $\{(0\ 0\ 1\ 0)^T, (2\ 0\ 0\ 0)^T, (1\ 0\ 0\ 1)^T, (0\ 2\ 0\ 0)^T, (0\ 0\ 0\ 2)^T\}$. N_4 is a member of the class $\mathcal{F}$. (e) The general FCPN structure N_5 belongs to the class $\mathcal{F}$. The weight associated with the arc from place p_1 to the uncontrollable transition t_2 (controllable transition t_1) is unity (two). Therefore, N_5 belongs to the class $\mathcal{F}$. The control-invariant, right-closed set of initial markings for which there is a supervisory policy that enforces liveness is defined by the minimal elements $\{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}$. (f) The general FCPN structure N_6 is not a member of the class $\mathcal{F}$. The weight associated with the arc from p_2 to the uncontrollable transition t_2 is not the smallest of the weights associated with arcs originating from p_2.

for any initial marking of this structure that belongs to the right-closed set whose minimal elements are $\{(0\ 0\ 1\ 0)^T, (2\ 0\ 0\ 0)^T, (1\ 0\ 0\ 1)^T, (0\ 2\ 0\ 0)^T, (0\ 0\ 0\ 2)^T\}$. The minimally restrictive supervisory policy that enforces liveness will disable the controllable transition t_5 at a marking if its firing would result in a new marking that is not in this right-closed set. As with the previous example the right-closed set of initial markings is *control-invariant*. That is, only the firing of the controllable transition t_5 could possibly result in a marking that is not in the aforementioned right-closed set.

The set of initial markings for which there is a supervisory policy that enforces liveness in the general FCPN structure N_5 shown in figure 1(e) is defined by the minimal elements $\{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}$. N_5 belongs to the class $\mathcal{F}$ as the outgoing arcs from p_3 to uncontrollable transitions t_5 and t_6 have a weight of unity; and the weight of the outgoing arc from p_1 to the uncontrollable transition t_2 is the smallest of all weights associated with arcs that originate from p_1.

We suggest explorations of other classes of general FCPNs where the set of initial markings for which there is a liveness-enforcing supervisory policy is right-closed, as a future research direction. For instance, the general FCPN structure N_6 shown in figure 1(f) does not belong to the class $\mathcal{F}$. This is because there are two outgoing arcs originating from p_1 that terminate on an uncontrollable transitions. Of these, the arc weight associated with (p_1, t_2) is not the smallest of all outgoing arc weights from p_1. That said, this general FCPN is live for any non-zero initial marking – that is, the set of initial markings for which there is a supervisory policy that enforces liveness in this general FCPN is indeed right-closed. As we note in subsequent text, this general FCPN does not confirm to any of the known sufficient conditions for liveness of general FCPN in the literature.

We also note that in all examples from the class $\mathcal{F}$, if there is a liveness enforcing supervisory policy, then there is a marking monotone supervisory policy that enforces liveness. That is, if a controllable transition is permitted to fire at a marking, then it is permitted to fire at any larger marking. We will see that this is not necessarily true of supervisory policies that enforce liveness in general FCPNs that do not belong to the class $\mathcal{F}$.

The remainder of the paper presents the theoretical underpinnings behind the observations regarding the right-closure of the set of initial markings for which there is a liveness enforcing supervisory policy for an arbitrary member of $\mathcal{F}$, along with a proof of the claim that the existence of a liveness-enforcing supervisory policy for any member of $\mathcal{F}$ is decidable. Section II presents the notations and definitions that are used in the remainder of the paper. This section also reviews the results in the literature that are relevant to this paper. The main results are presented in section III, where it is shown that the proof of similar claims made for ordinary FCPNs in reference [5] apply *mutatis mutandis* to the class $\mathcal{F}$ defined in this paper. We conclude with some suggested directions for future research in section IV.

II. Notations and Definitions and Some Preliminary Observations

We use $\mathcal{N}$ ($\mathcal{N}^+$) to denote the set of non-negative (positive) integers. A *Petri net structure* $N = (\Pi, T, \Phi, \Gamma)$ is an ordered 4-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, \ldots, t_m\}$ is a collection of m *transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, and $\Gamma : \Phi \rightarrow \mathcal{N}^+$ is the *weight* associated with each arc. The *initial marking function* (or the *initial marking*) of a PN structure N is a function $\mathbf{m}^0 : \Pi \rightarrow \mathcal{N}$, which identifies the number of *tokens* in each place. We will use the term *Petri net* (PN) to denote a PN structure along with its initial marking $\mathbf{m}^0$, and is denoted by the symbol $N(\mathbf{m}^0)$.

A marking $\mathbf{m} : \Pi \rightarrow \mathcal{N}$ is sometimes represented by an integer-valued vector $\mathbf{m} \in \mathcal{N}^n$, where the i-th component $\mathbf{m}_i$ represents the token load ($\mathbf{m}(p_i)$) of the i-th place. Extending this notation to integer-valued vectors in general, the i-th component of any integer valued vector $\mathbf{x}$ is denoted by $\mathbf{x}_i$. The function- and vector-interpretation of the marking is used interchangeably in this paper. The context should indicate the appropriate interpretation.

In graphical representation of PNs places (transitions) are represented by circles (boxes), and each member of $\phi \in \Phi$ is denoted by a directed arc. If $\phi = (p, t)$ ((t, p)) the arc is directed from p (t) to t (p). The weight of the arc, $\Gamma(\phi)$ is represented by an integer that is placed along side the arc. For brevity, we refrain from denoting the weight of those arcs $\phi \in \Phi$ where $\Gamma(\phi) = 1$. The initial marking is represented by $\mathbf{m}^0(p)$-many filled-circles, or tokens, within each place $p \in \Pi$. The sets ${}^\bullet x := \{y \mid (y, x) \in \Phi, \text{where } N = (\Pi, T, \Phi)\}$ and $x^\bullet := \{y \mid (x, y) \in \Phi, \text{where } N = (\Pi, T, \Phi)\}$ will find use in subsequent text.

For a given marking $\mathbf{m}^i$, a transition $t \in T$ is said to be *enabled* if $\forall p \in ({}^\bullet t)_N, \mathbf{m}^i(p) \geq \Gamma((p, t))$. The set of enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$. An enabled transition $t \in T_e(N, \mathbf{m}^i)$ can *fire*, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$ according to the equation $\mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \Gamma((p, t)) + \Gamma((t, p))$.

A string of transitions $\sigma = t_1 t_2 \cdots t_k$, where $t_j \in T (j \in \{1, 2, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$, if, (1) the transition $t_1 \in T_e(N, \mathbf{m}^i)$, and (2) for $j \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ is enabled. If $\mathbf{m}^{i+k}$ results from the firing of $\sigma \in T^*$ starting from the initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \rightarrow \sigma \rightarrow \mathbf{m}^{i+k}$. Given an initial marking $\mathbf{m}^0$ the set of *reachable markings* for $\mathbf{m}^0$ denoted by $\mathfrak{R}(N, \mathbf{m}^0)$, is defined as the set of markings generated by all valid firing strings starting with marking $\mathbf{m}^0$ in the PN structure N. A PN $N(\mathbf{m}^0)$ is said to be *live* if $\forall t \in T, \forall \mathbf{m}^i \in \mathfrak{R}(N, \mathbf{m}^0), \exists \mathbf{m}^j \in \mathfrak{R}(N, \mathbf{m}^i)$ such that $t \in T_e(N, \mathbf{m}^j)$.

In those contexts where the marking is interpreted as a nonnegative integer-valued vector, it is useful to define the *input matrix* **IN** and *output matrix* **OUT** as two $n \times m$ matrices, where

$$\mathbf{IN}_{i,j} = \begin{cases} \Gamma((p_i, t_j)) & \text{if } p_i \in {}^\bullet t_j, \\ 0 & \text{otherwise,} \end{cases} \text{ and } \mathbf{OUT}_{i,j} = \begin{cases} \Gamma((t_j, p_i)) & \text{if } p_i \in t_j^\bullet, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The *incidence matrix* $\mathbf{C}$ of the PN N is an $n \times m$ matrix, where $\mathbf{C} = \mathbf{OUT} - \mathbf{IN}$. If $\mathbf{x}(\sigma)$ is an m-dimensional vector whose i-th component corresponds to the number of occurrences of t_i in a valid string $\sigma \in T^*$, and if $\mathbf{m}^i \rightarrow \sigma \rightarrow \mathbf{m}^{i+j}$, then $\mathbf{m}^{i+j} = \mathbf{m}^i + \mathbf{C}\mathbf{x}(\sigma)$.

A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \geq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 \in \mathcal{M}))$. Every right-closed set of vectors $\mathcal{M} \subseteq \mathcal{N}^n$ contains a finite set of minimal-elements, $min(\mathcal{M}) \subset \mathcal{M}$, such that (i) $\forall \mathbf{m}^1 \in \mathcal{M}, \exists \mathbf{m}^2 \in min(\mathcal{M})$, such that $\mathbf{m}^1 \geq \mathbf{m}^2$, and (ii) if $\exists \mathbf{m}^1 \in \mathcal{M}, \exists \mathbf{m}^2 \in min(\mathcal{M})$ such that $\mathbf{m}^2 \geq \mathbf{m}^1$, then $\mathbf{m}^1 = \mathbf{m}^2$. In general the (finite) set of minimal elements $min(\mathcal{M})$ of a right-closed set $\mathcal{M}$ might not be effectively computable. Valk and Jantzen [8] present a necessary and sufficient condition that guarantees the effective computability of $min(\mathcal{M})$ for an arbitrary right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$.

A collection of places $P \subseteq \Pi$ is said to be a *siphon* (*trap*) if ${}^\bullet P \subseteq P^\bullet$ ($P^\bullet \subseteq^\bullet P$), where ${}^\bullet P := \bigcup^\bullet_{p \in P} p$ and $P^\bullet := \bigcup_{p \in P} p^\bullet$. A trap (siphon) P, is said to be *minimal* if $\nexists \widetilde{P} \subset P$, such that $\widetilde{P}^\bullet \subseteq^\bullet \widetilde{P}$ (${}^\bullet\widetilde{P} \subseteq \widetilde{P}^\bullet$). A PN structure $N = (\Pi, T, \Phi)$ is a *Free-Choice* if $\forall p \in \Pi, (card(p^\bullet) > 1 \Rightarrow ({}^\bullet(p^\bullet) = \{p\})$. In other words, a PN structure is Free-Choice if and only if an arc from a place to a transition is either the unique output arc from that place, or, is the unique input arc to the transition. A PN $N(\mathbf{m}^0)$ where $N = (\Pi, T, \Phi)$ is free choice, is said to be a *Free-Choice Petri net* (FCPN). A general PN structure $N = (\Pi, T, \Phi)$ is said to belong to the class $\mathcal{F}$ if,

1) N is an FCPN structure, and
2) $\forall p \in \Pi$,

$$((p^\bullet \cap T_u \neq \emptyset) \wedge (card(p^\bullet) > 1)) \Rightarrow \left(\forall t_u \in p^\bullet \cap T_u, \Gamma((p, t_u) = \min_{\phi \in (\{p\} \times T) \cap \Phi} \Gamma(\phi)\right). \tag{2}$$

That is, $\mathcal{F}$ is a collection of general FCPNs such that if place in a member of $\mathcal{F}$ has multiple output transitions, and at least one of them is uncontrollable, then the weights associated with every arc from the place at hand to any uncontrollable transition must be equal to the smallest weight of among all arcs originating from the place. Alternately, if $N \in \mathcal{F}$,

$$\forall \mathbf{m}^i \in \mathfrak{R}(N, \mathbf{m}^0), (T_e(N, \mathbf{m}^i) \cap p^\bullet \neq \emptyset) \Rightarrow ((p^\bullet \cap T_u) \subseteq T_e(N, \mathbf{m}^i)).$$

The ordinary FCPN structure N_1 shown in figure 1(a) belongs to the class $\mathcal{F}$, as every arc has a unit weight associated with it. The general FCPN structure N_2 shown in figure 1(b) does not belong to the class $\mathcal{F}$ as $\Gamma((p_5, t_6)) = 2$, while $\min\{\Gamma((p_5, t_5)), \Gamma((p_5, t_6))\} = 1$ and transition t_6 is uncontrollable. The general FCPN structures N_3 and N_4 shown in figures 1(c) and 1(d) belong to the class $\mathcal{F}$ as all outgoing arcs from a place to a transition in these structures have the same weight of two. The general FCPN structure N_5 shown in figure 1(e) belongs to $\mathcal{F}$ as both the outgoing arcs from place p_3 have a weight of unity, and the weight associated with the arc (p_1, t_2) is the smallest of all weight associated with arcs that originate from p_1. The general FCPN structure N_6 shown in figure 1(f) does not belong to the class $\mathcal{F}$ as $\Gamma((p_1, t_2)) = 2$ and $\min\{\Gamma((p_1, t_2)), \Gamma((p_1, t_3))\} = 1$, and transition t_2 is uncontrollable.

A. Supervisory Control of PNs

The paradigm of supervisory control of PNs assumes a subset of *controllable transitions*, denoted by $T_c \subseteq T$, can be prevented from firing by an external agent called the *supervisor*. The set of *uncontrollable transitions*, denoted by $T_u \subseteq T$, is given by $T_u = T - T_c$. The controllable (uncontrollable) transitions are represented as filled (unfilled) boxes in graphical representation of PNs.

A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \times T \rightarrow \{0, 1\}$, is a function that returns a 0 or 1 for each transition and each reachable marking. The supervisory policy $\mathcal{P}$ permits the firing of transition t_j at marking $\mathbf{m}^i$, only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $t_j \in T_e(N, \mathbf{m}^i)$ for some marking $\mathbf{m}^i$, we say the transition t_j is *state-enabled* at $\mathbf{m}^i$. If $\mathcal{P}(\mathbf{m}^i, t_j) = 1$, we say the transition t_j is *control-enabled* at $\mathbf{m}^i$. A transition has to be state- and control-enabled before it can fire. The fact that uncontrollable transitions cannot be prevented from firing by the supervisory policy is captured by the requirement that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$. This is implicitly assumed of any supervisory policy in this paper.

A string of transitions $\sigma = t_1 t_2 \cdots t_k$, where $t_j \in T (j \in \{1, 2, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$, if,

1) $t_1 \in T_e(N, \mathbf{m}^i), \mathcal{P}(\mathbf{m}^i, t_1) = 1$, and
2) for $j \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{j+1}) = 1$.

The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$. A supervisory policy $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$ is said to be *marking monotone*, if $\forall t \in T, \forall \{\mathbf{m}^j, \mathbf{m}^i\} \subseteq \mathcal{N}^n, (\mathbf{m}^j \geq \mathbf{m}^i) \Rightarrow (\mathcal{P}(\mathbf{m}^j, t) \geq \mathcal{P}(\mathbf{m}^i, t))$. That is, if a transition is control-enabled at some marking by a marking monotone policy, it remains control-enabled for all larger markings. A transition t_k is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$. A supervisory policy $\mathcal{P}$ enforces liveness if all transitions in $N(\mathbf{m}^0)$ are live under $\mathcal{P}$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if for every supervisory policy $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \to \{0, 1\}$ that enforces liveness in $N(\mathbf{m}^0)$, the following condition holds

$$\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t).$$

Alternately, if a minimally restrictive supervisory policy $\mathcal{P}$ that enforces liveness in $N(\mathbf{m}^0)$ prevents the occurrence of transition $t \in T$ at some marking $\mathbf{m}^i \in \mathcal{N}^n$, then every policy that enforces liveness in $N(\mathbf{m}^0)$ should prevent the occurrence of $t \in T$ for the marking $\mathbf{m}^i$. There is a unique minimally restrictive policy that enforces liveness in every PN $N(\mathbf{m}^0)$ that has some policy that enforces liveness (cf. theorem 6.1, [6]). The existence of a supervisory policy that enforces liveness in an arbitrary PN is undecidable [6], and is decidable if all transitions in the PN are controllable [6], or if we restricted attention to *ordinary* FCPNs where $\Gamma(\phi) = 1, \forall \phi \in \Phi$ [5]. This follows from the fact that if there is a supervisory policy that enforces liveness in an arbitrary ordinary FCPN $N(\mathbf{m}^0)$, then there is a supervisory policy that enforces liveness in $N(\widetilde{\mathbf{m}}^0)$ for any $\widetilde{\mathbf{m}}^0 \geq \mathbf{m}^0$. That is, the set of initial markings, $\Delta(N)$, for which there is a supervisory policy that enforces liveness in an arbitrary ordinary FCPN is *right-closed*[1] and *control-invariant*[2], and is uniquely identified by its minimal elements. The control-invariance of a right-closed set of markings is decidable (cf. Lemma 5.10, [5]). The minimally restrictive supervisory policy that enforces liveness ensures the marking of the ordinary FCPN does not leave this right-closed, control-invariant set of markings.

Suppose $\Delta_f(N)$ denotes the right-closed, control-invariant set of initial markings for which there is a supervisory policy that enforces liveness in a version of the arbitrary ordinary FCPN where all transitions in N are assumed to be controllable, it follows that $\Delta(N) \subseteq \Delta_f(N)$. The minimal elements that define the set $\Delta_f(N)$ are effectively computable. The semi-decidability of the existence of a supervisory policy that enforces liveness in an arbitrary ordinary FCPN can be established by a brute-force enumeration of the minimal elements of right-closed, control-invariant sets till those that identify the set $\Delta(N) \subseteq \Delta_f(N)$ are found. The semi-decidability of the non-existence of a supervisory policy that enforces liveness in an arbitrary ordinary FCPN is established by a process of iteratively identifying right-closed subsets, $\{\Delta_i(N)\}_{i=1}^{\infty}$, such that $\Delta_f(N) \supset \Delta_1(N) \supset \cdots \supset \Delta_i(N) \supset \cdots$, and no member of $\Delta_f(N) - \Delta_i(N)$ can be in a control-invariant subset of $\Delta_f(N)$. The particulars of this iterative process ensures that when $\mathbf{m}^0 \notin \Delta(N)$, $\mathbf{m}^0 \notin \Delta_i(N)$ for an appropriate value of i, which results in termination of the iteration. These two semi-decidability results together imply the decidability of the existence of a supervisory policy that enforces liveness in an arbitrary ordinary FCPN.

[1]A set of markings $\Omega \subseteq \mathcal{N}^n$ is *right-closed* if $(\mathbf{m}_1 \in \Omega) \wedge (\mathbf{m}_2 \geq \mathbf{m}_1) \Rightarrow (\mathbf{m}_2 \in \Omega)$

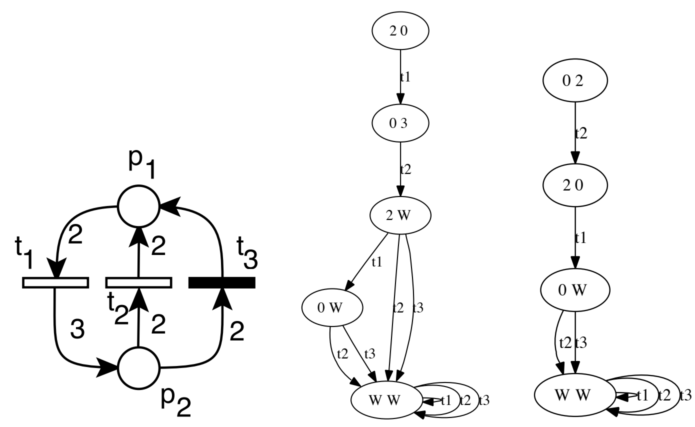
[2]A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *control-invariant* [9] with respect to a partially controlled PN structure $N = (\Pi, T, \Phi)$, if $\mathcal{M} = \Gamma(\mathcal{M})$, where $\Gamma(\mathcal{M}) = \{\mathbf{m}^i \in \mathcal{N}^n \mid \exists \sigma \in T_u^*, \exists \mathbf{m}^j \in \mathcal{M}, \text{ such that } \mathbf{m}^j \to \sigma \to \mathbf{m}^i\}$

If there is a supervisory policy that enforces liveness in an ordinary FCPN, then there is a marking monotone supervisory policy that enforces liveness in the ordinary FCPN. The process of deciding the existence of a supervisory policy that enforces liveness in an arbitrary ordinary FCPN is NP-hard.

The procedure shown in figure 1 of reference [5] can be used to generate the *KM-Tree* of a general PN $N(\mathbf{m}^0)$ under the influence of a marking monotone supervisory policy $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$. We get the *coverability graph*, $G(N, \mathbf{m}^0, \mathcal{P})$, when we merge the duplicate vertices of the KM-Tree of a general PN as one. The properties of the coverability graph that were used in various proofs in reference [5] in the context of ordinary PNs, apply to general PNs as well. As an illustration, figure 2(a) (figure 2(b)) shows the coverability graph of the general FCPN N_3 shown in figure 1(c) with an initial marking of $(2\ 0)^T$ ($(0\ 2)^T$) under the marking monotone supervisory policy $\mathcal{P}_3 : \mathcal{N}^2 \times T_3 \rightarrow \{0, 1\}$ which disables the controllable transition t_3 whenever its firing results in a marking that is not greater than or equal to any one the two minimal elements in the set $\{(2\ 0)^T, (0\ 2)^T\}$.

A vertex in the coverability graph is identified by the extended marking associated with it. A path in the coverability graph is represented notationally as $v_i \rightarrow \sigma \rightarrow v_j$, where v_i (v_j) is the origin (terminus) of the path that is uniquely identified by a sequence of transition labels. The existence of the path $(2\ 0)^T \rightarrow t_1t_2 \rightarrow (2\ \omega)^T$ in $G(N_3, (2\ 0)^T, \mathcal{P}_3)$, implies that for any $k \in \mathcal{N}, \exists l \in \mathcal{N}$ such that $(2\ 0)^T \rightarrow (t_1t_2)^l \rightarrow (2\ \alpha)^T$, where $\alpha \geq k$. That is, the token load of p_2 can be made larger than any fixed integer l with an appropriate repetitions of the firing string t_1t_2.

In subsequent text we will seek closed-paths in coverability graphs like $(\omega\ \omega)^T \rightarrow t_2t_1t_2t_1t_3 \rightarrow (\omega\ \omega)^T$ in $G(N_3, (2\ 0)^T, \mathcal{P}_3)$, for example. The key aspects of closed-paths such as this are (1) all transitions in the PN appear at least once in the path, and (2) the marking that results at the end of this firing string is greater than or equal to the marking before the transitions in the path are fired (for example, $\mathbf{C}_3\mathbf{x}(t_2t_1t_2t_1t_3) = (1\ 0)^T \geq \mathbf{0}$, where $\mathbf{C}_3$ is the incidence matrix of N_3 and $\mathbf{0}$ is the vector of all zeros). Since there is a path in $G(N_3, (2\ 0)^T, \mathcal{P}_3)$ from the initial vertex to the vertex labeled with the extended marking $(\omega\ \omega)^T$, it stands to reason that there is a valid firing string from the initial marking $(2\ 0)^T$ to a marking from which the string $t_2t_1t_2t_1t_3$ can be repeated as often as necessary. For example, in this case $(2\ 0)^T \rightarrow t_1t_2t_1t_2 \rightarrow (2\ 2)^T \rightarrow t_2t_1t_2t_1t_3 \rightarrow (3\ 2)^T \rightarrow t_2t_1t_2t_1t_3 \rightarrow \cdots$; and the firing string $t_2t_1t_2t_1t_3$ can be repeated as often as necessary.



(a) General FCPN structure N_3 (b) $G(N_3, (2\ 0)^T), \mathcal{P}_3$ (c) $G(N_3, (0\ 2)^T), \mathcal{P}_3$

Fig. 2. (a) The general FCPN N_3 of figure 1(c). (b) The coverability graph of N_3 when initialized with the marking $(2\ 0)^T$, under the influence of the policy $\mathcal{P}_3$. (c) The coverability graph of N_3 when initialized with the marking $(0\ 2)^T$, under the influence of the policy $\mathcal{P}_3$.

B. Review of Relevant Prior Work

Commoner's Liveness Theorem (cf. [2]; chapter 4, [10]) states an ordinary FCPN $N(\mathbf{m}^0)$ is live if and only if every minimal siphon in N contains a minimal trap that has a non-empty token load at the initial marking $\mathbf{m}^0$. Testing the liveness of an ordinary FCPN is NP-hard (cf. Problem MS3, [11]). The ordinary FCPN shown in figure 1(a) is not live in the absence of supervision for any initial marking, as the siphon $\{p_1, p_2, p_3, p_4\}$ does not contain any traps. As noted in the introductory section, the ordinary FCPN $N_1(\mathbf{m}_1^0)$ can be made live by supervision for any initial marking $\mathbf{m}_1^0$ that is greater than or equal to (at least) one of the members of the set $\{(1\ 0\ 0\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0\ 0\ 0)^T\}$.

A PN structure $N = (\Pi, T, \Phi)$ is said to be a *Simple Petri Net* (SPN) if and only if

$$\forall t \in T, card(\{p \in^{\bullet} t \mid card(p^{\bullet}) > 1\}) \leq 1.$$

The family of FCPN structures is strictly contained in the family of SPN structures. Barkaoui et al. [12] characterize the liveness of a general SPN $N(\mathbf{m}^0)$ in two ways. First, they note that if all siphons $P \subseteq \Pi({}^{\bullet}P \subseteq P^{\bullet})$ of $N(\mathbf{m}^0)$ satisfy the requirement – $\forall \mathbf{m} \in \Re(N, \mathbf{m}^0), \exists p \in P$ such that $\mathbf{m}(p)$ is greater than or equal to the largest weight among the arcs that originate from p – then $N(\mathbf{m}^0)$ is live. Second, they note that when the weights on all outgoing arcs from any place in $N(\mathbf{m}^0)$ are the same, then this condition is both necessary and sufficient for the liveness of $N(\mathbf{m}^0)$.

The weights associated with the arcs originating from p_1 in the general FCPN structure N_6 shown in figure 1(f) are not the same. So, the second characterization of liveness is not applicable for this general FCPN structure. If this structure is initialized with the marking $\mathbf{m}_1^0 = (1\ 0\ 0\ 0\ 0)^T$, then $\mathbf{m}_1^0(p_1) = 1$, which is not greater than or equal to the largest weight (which is two) of all arcs originating from p_1. However, N_6 is live for any non-zero initial marking. That is, the converse of the first characterization is not true in general.

Similarly, the second characterization of liveness is inapplicable to the general FCPN structure N_2 shown in figure 1(b) as the weights associated with all output arcs from a place are not necessarily identical. The FCPN $N_2(\mathbf{m}_2^0)$ is live for any $\mathbf{m}_2^0$ such that $\mathbf{1}^T\mathbf{m}_2^0$ is an odd number, where $\mathbf{1}$ is the vector of all ones. The marking $\mathbf{m}_2^i$ that would result from the firing of a valid firing string $\sigma \in T_2^*$ at the initial marking $\mathbf{m}_2^0$ is given by the expression

$$\mathbf{m}_2^i = \mathbf{m}_2^0 + \mathbf{C}_2\mathbf{x}(\sigma) \Rightarrow \mathbf{1}^T\mathbf{m}_2^i = \mathbf{1}^T\mathbf{m}_2^0 + \mathbf{1}^T\mathbf{C}_2\mathbf{x}(\sigma) = \mathbf{1}^T\mathbf{m}_2^0 + (2 \times (\#(\sigma, t_1) - \#(\sigma, t_6))),$$

where $\#(\sigma, t_i)$ denotes the number of occurrences of the transition t_i in the string σ. Therefore, if $\mathbf{1}^T\mathbf{m}_2^0$ is an odd (positive) number, then $\mathbf{1}^T\mathbf{m}_2^i$ is an odd (positive) number. That is, $\mathbf{1}^T\mathbf{m}_2^i$ is non-zero. Since all transitions in this general FCPN are potentially fireable if there is at least one token in some place, we conclude that this general FCPN is live under the stipulated condition. It can also be argued that the zero-making is reachable if $\mathbf{1}^T\mathbf{m}_2^0$ is an even number, which implies the general FCPN is not live if $\mathbf{1}^T\mathbf{m}_2^0$ is an even number. This is in stark contrast to the property of liveness monotonicity seen in ordinary FCPNs, which is a consequence of Commoner's Liveness Theorem.

Guia [13] introduced *monitors* into supervisory control of PNs. Monitors are external places added to an existing PN structure whose token load at any instant indicates the amount of a particular resource that is available for consumption. This concept was used by Moody and Antsaklis, who used *monitor-based supervisors* to enforce liveness in certain classes of PNs [14], which was then extended by Iordache and Antsaklis [15] to include a sufficient condition for the existence of policies that enforce liveness in a class of PNs called *Asymmetric Choice Petri nets*[3]. Reveliotis [16] developed a class of policies for resource allocation systems that can be extended to the PN-framework using the *theory of regions* [17]. Ghaffari, Rezg and Xie [18] also use the theory of regions to obtain a *maximally permissive* supervisory policy that enforces liveness for a class of PNs.

In the next section we show that the proofs of various claims made in reference [5] for ordinary FCPNs, with appropriate changes, serve as a proof of similar claims for general FCPNs that belong to the class $\mathcal{F}$. Specifically, for $N \in \mathcal{F}$, the set of initial markings for which there is a liveness enforcing supervisory policy, $\Delta(N)$, is right-closed. This is followed by other observations which eventually lead to the conclusion that the existence of a liveness enforcing supervisory policy for any member of $\mathcal{F}$ is

[3]cf. page 554, [4] for a formal definition.

decidable. In addition, the minimally restrictive supervisory policy that enforces liveness in any member of $\mathcal{F}$, assuming it exists, is a marking monotone policy that effectively maintains the marking of the FCPN to remain within the right-closed set of initial markings $\Delta(N)$.

III. Main Results

For any arbitrary PN structure $N = (\Pi, T, \Phi, \Gamma)$ where $T = T_u \cup T_c (T_c \cap T_u = \emptyset)$, we define the set of initial markings for which there is a liveness enforcing supervisory policy as

$$\Delta(N) = \{\mathbf{m}^0 \in \mathcal{N}^{card(\Pi)} \mid \exists \text{ a liveness enforcing supervisory policy for } N(\mathbf{m}^0)\}. \tag{3}$$

If $\mathbf{m}^0 \in \Delta(N)$, there is a policy $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ that enforces liveness in $N(\mathbf{m}^0)$. Suppose $\mathbf{m}^0 \in \Delta(N)$ and $\mathbf{m}^0 \rightarrow t_u \rightarrow \mathbf{m}^1$ in N from some $t_u \in T_u$, since all uncontrollable transitions are permanently control-enabled, it follows that $\mathbf{m}^0 \rightarrow t_u \rightarrow \mathbf{m}^1$ in N under the supervision of $\mathcal{P}$. This in turn implies that $\mathbf{m}^1 \in \Delta(N)$. This leads to the following characterization of $\Delta(N)$ for any arbitrary N.

Lemma 3.1: For any arbitrary PN structure $N = (\Pi, T, \Phi, \Gamma)$ the set $\Delta(N)$ (cf. equation 3) is control-invariant.

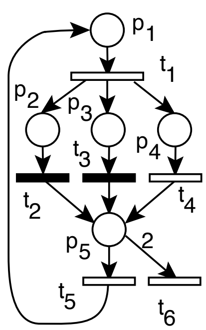
As an illustration of lemma 3.1, consider the set $\Delta(N_2)$ for the general FCPN N_2 shown in figure 1(b), where

$$\Delta(N_2) = \{\mathbf{m}_2^0 \in \mathcal{N}^5 \mid (\mathbf{1}^T \mathbf{m}_2^0)_{mod\ 2} = 1\},$$

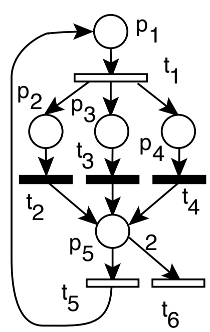
and $\mathbf{1}$ is the vector of all ones. Suppose $\mathbf{m}_2^1 \in \Delta(N_2)$ and $\mathbf{m}_2^1 \rightarrow t_u \rightarrow \mathbf{m}_2^2$ for some $t_u \in \{t_1, t_2, \ldots, t_6\}$. By taking each possible instance of t_u, we can show that $(\mathbf{1}^T \mathbf{m}_2^2)_{mod\ 2} = 1$. That is, $\mathbf{m}_2^2 \in \Delta(N_2)$, which in turn implies its control-invariance. Since all transitions in N_2 are potentially fireable as long as there is at least one token in some place in N_2, it follows that $\Delta(N_2)$ is the set of initial markings for which there is a (trivial; as all transitions in N_2 are uncontrollable) liveness enforcing supervisory policy. Unlike the general FCPN N_2 of figure 1(b), transitions t_2 and t_3 (t_2, t_3 and t_4) are controllable in the general FCPN structure N_7 (N_8) shown in figure 3(a) (figure 3(b)). The sets $\Delta(N_7)$ and $\Delta(N_8)$ are defined below.

$$\begin{aligned}
\Delta(N_7) &= \{\mathbf{m}^0 \in \mathcal{N}^5 \mid (\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) \geq 1) \vee ((\mathbf{m}(p_4) + \mathbf{m}(p_5))_{mod\ 2} = 1)\} \\
\Delta(N_8) &= \{\mathbf{m}^0 \in \mathcal{N}^5 \mid (\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_4) \geq 1) \vee ((\mathbf{m}(p_5))_{mod\ 2} = 1)\}
\end{aligned}$$

The firing of uncontrollable transitions t_1, t_4, t_5 and t_6 (t_1, t_5 and t_6) at any marking in the set $\Delta(N_7)$ ($\Delta(N_8)$) will result in a new marking that belongs to the set $\Delta(N_7)$ ($\Delta(N_8)$), which establishes the control-invariance of the set $\Delta(N_7)$ ($\Delta(N_8)$) with respect to the structure N_7 (N_8).



(a) N_7



(b) N_8

Fig. 3. (a) The general FCPN N_7. (b) The general FCPN N_8.

There is a supervisory policy that enforces liveness in an arbitrary general PN $N(\mathbf{m}^0)$ where $N = (\Pi, T, \Phi, \Gamma)$, if and only if $\mathbf{m}^0 \in \Delta(N)$ (cf. equation 3). Assuming $\mathbf{m}^0 \in \Delta(N)$, we define a supervisory policy $\widetilde{\mathcal{P}} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ as follows

$$\left(\widetilde{\mathcal{P}}(\mathbf{m}^i, t_j) = 0\right) \Leftrightarrow \left(t_j \in T_e(N, \mathbf{m}^i), \mathbf{m}^i \rightarrow t_j \rightarrow \mathbf{m}^k \text{ in } N, \text{ and } \mathbf{m}^k \notin \Delta(N)\right). \tag{4}$$

Since $\Delta(N)$ is control-invariant, all uncontrollable transitions are permanently control-enabled for all markings under $\widetilde{\mathcal{P}}$. The following theorem identifies the supervisory policy $\widetilde{\mathcal{P}}$ as the minimally restrictive supervisory policy that enforces liveness in an arbitrary PN $N(\mathbf{m}^0)$, assuming $\mathbf{m}^0 \in \Delta(N)$.

Theorem 3.2: Let $N(\mathbf{m}^0)$ be an arbitrary PN, where $\mathbf{m}^0 \in \Delta(N)$, then the supervisory policy $\widetilde{\mathcal{P}}$ identified in equation 4 is the minimally restrictive supervisory policy that enforces liveness in $N(\mathbf{m}^0)$.

Proof: The policy $\widetilde{\mathcal{P}}$ prevents the firing a state-enabled, controllable transition only if its firing at a marking would result in a new marking that does not belong to $\Delta(N)$. So, if $\mathcal{P}$ is a supervisory policy that enforces liveness in liveness in $N(\mathbf{m}^0)$, then $\Re(N, \mathbf{m}^0, \mathcal{P}) \subseteq \Re(N, \mathbf{m}^0, \widetilde{\mathcal{P}})$.

If $\mathbf{m}^0 \to \sigma_1 \to \mathbf{m}^1$ under $\widetilde{\mathcal{P}}$ in N, then there is a supervisory policy $\mathcal{P}$ that enforces liveness in $N(\mathbf{m}^1)$. If $\mathbf{m}^1 \to \sigma_2 \to \mathbf{m}^2$ under $\mathcal{P}$ in $N(\mathbf{m}^1)$, then using fact that $\Re(N, \mathbf{m}^0, \mathcal{P}) \subseteq \Re(N, \mathbf{m}^0, \widetilde{\mathcal{P}})$, and an induction argument over the length of σ we can show that $\mathbf{m}^1 \to \sigma_2 \to \mathbf{m}^2$ under $\widetilde{\mathcal{P}}$ in $N(\mathbf{m}^1)$ too. This establishes the fact that $\widetilde{\mathcal{P}}$ is the minimally restrictive policy that enforces liveness in $N(\mathbf{m}^0)$. ■

From theorem 3.2, and the definition of $\Delta(N)$, we note there is a supervisory policy that enforces liveness in an arbitrary general PN $N(\mathbf{m}^0)$ if and only if membership in $\Delta(N)$ is decidable. From theorem 3.1 (theorem 3.2) of reference [5] we conclude that there is no positive-test for membership (non-membership) in $\Delta(N)$ for an arbitrary general PN structure N.

The results in reference [5] suggest that the right-closure of $\Delta(N)$ is particularly useful for deciding membership in $\Delta(N)$ for certain classes of PN structures. Specifically, $\Delta(N)$ is right-closed for the class of ordinary FCPNs, which eventually leads to the conclusion that the existence of a liveness enforcing supervisory policy in an a arbitrary ordinary FCPN is decidable [5]. Unfortunately, this property does not hold for the class of general FCPNs. For example. the sets $\Delta(N_2), \Delta(N_7)$ and $\Delta(N_8)$, in the general FCPN examples shown in figures 1(b), 3(a) and 3(b) are not right-closed.

Using a string of observations, we show that $\Delta(N)$ is right-closed for general FCPN structures that belong to the class $\mathcal{F}$. A general FCPN is said to belong to the class $\mathcal{F}$ if and only if the weight associated with an arc from a place to an uncontrollable transition is the smallest of the weights associated with all arcs originating from the place. The following result is the parallel of lemma 5.1 in reference [5] originally stated for ordinary FCPNs, for general FCPNs that belong to the class $\mathcal{F}$.

Lemma 3.3: Let $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \to \{0, 1\}$ be supervisory policy that enforces liveness in a general FCPN $N(\mathbf{m}^0)$, where $N = (\Pi, T, \Phi, \Gamma)$, $T = T_c \cup T_u$, and $N \in \mathcal{F}$. Suppose $\mathbf{m}^0 \to \sigma \to \mathbf{m}^i$ under the supervision of $\mathcal{P}$ in N, and $\widehat{\mathbf{m}}^0 \to \widehat{\sigma} \to \widehat{\mathbf{m}}^j$ in the absence of any supervision in N for some $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$. Further, let us suppose that the number of occurrences of each controllable transition in $\widehat{\sigma}$ are σ are identical; however, the string $\widehat{\sigma}$ has a few more uncontrollable transitions than the string σ. That is, $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \subseteq T_u$. Then $\exists \widetilde{\sigma}_1, \widetilde{\sigma}_2 \in T^*$, such that

1) $\widehat{\mathbf{m}}^j \to \widetilde{\sigma}_1 \to \widehat{\mathbf{m}}^k$ in N in the absence of any supervision,
2) $\mathbf{m}^i \to \widetilde{\sigma}_2 \to \mathbf{m}^l$ under the supervision of $\mathcal{P}$ in N, and
3) $\mathbf{x}(\widehat{\sigma}\widetilde{\sigma}_1) = \mathbf{x}(\sigma\widetilde{\sigma}_2)$ $(\Rightarrow \widehat{\mathbf{m}}^k \geq \mathbf{m}^l)$.

That is, $\mathbf{m}^0 \to \sigma\widetilde{\sigma}_2 \to \mathbf{m}^l$ under the supervision of $\mathcal{P}$, and $\widehat{\mathbf{m}}^0 \to \widehat{\sigma}\widetilde{\sigma}_1 \to \widehat{\mathbf{m}}^k$ in the absence of any supervision in N. If $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$ and $\mathbf{x}(\sigma\widetilde{\sigma}_2) = \mathbf{x}(\widehat{\sigma}\widetilde{\sigma}_1)$, then $\widehat{\mathbf{m}}^k \geq \mathbf{m}^l$.

Proof: Since $\mathcal{P}$ enforces liveness in $N(\mathbf{m}^0)$, we can pick a string $\sigma_1 \in T^*$ such that

1) $\mathbf{m}^i \to \sigma_1 \to \mathbf{m}^{i+1}$ under $\mathcal{P}$ in N,
2) $\forall \overline{\sigma}_1 \in pr(\sigma_1) - \{\sigma_1\}$, if $\mathbf{m}^i \to \overline{\sigma}_1 \to \overline{\mathbf{m}}$, then $T_e(N, \overline{\mathbf{m}}) \cap \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} = \emptyset$, where $pr(\bullet)$ is the prefix-set of the string argument, and
3) $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \cap T_e(N, \mathbf{m}^{i+1}) \neq \emptyset$.

That is, none of the transitions in the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \subseteq T_u$ are state-enabled (and trivially control-enabled) following the firing of any proper prefix of the firing string σ_1. Additionally, at least one member of the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$ is state-enabled (and trivially control-enabled) at the marking $\mathbf{m}^{i+1}$ that results from the firing of the string σ_1 at $\mathbf{m}^i$.

It follows that $\widehat{\mathbf{m}}^j \to \sigma_1 \to \widehat{\mathbf{m}}^{j+1}$ in the absence of any supervision in N, which can be established by contradiction. Suppose $\sigma_1 = \bar{t}_1 \cdots \bar{t}_i \bar{t}_{i+1} \cdots$, and $\widehat{\mathbf{m}}^j \to \bar{t}_1 \cdots \bar{t}_i \to \widehat{\mathbf{m}}^{j+2}$, but $\bar{t}_{i+1} \notin T_e(N, \widehat{\mathbf{m}}^{j+2})$. This must be due to the reduction in the number of tokens in an input place of $\bar{t}_{i+1}$, as a result of the firing of some transition $t_u \in \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} (\subseteq T_u)$. Since N is an FCPN structure, transitions t_u and $\bar{t}_{i+1}$ must share a unique input place (i.e. $^\bullet t_u \cap^\bullet \bar{t}_{i+1} = \{p\}$ for some $p \in \Pi$). Since $N \in \mathcal{F}$, whenever p has sufficient tokens to state-enable $\bar{t}_{i+1}$, it follows that t_u is also state-enabled at the same marking. This contradicts the second of three conditions required of σ_1.

If $t_u \in \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \cap T_e(N, \mathbf{m}^{i+1})$, then $\mathbf{m}^0 \rightarrow \sigma\sigma_1 \rightarrow \mathbf{m}^{i+1} \rightarrow t_u \rightarrow \mathbf{m}^{i+2}$ under $\mathcal{P}$ in N. As noted above, $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma}\sigma_1 \rightarrow \widehat{\mathbf{m}}^{j+1}$ in the absence of any supervision in N. Additionally, $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma}\sigma_1)_j > \mathbf{x}(\sigma\sigma_1 t_u)_j\} \subset \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$. The claim is established by replacing σ with $\sigma\sigma_1 t_u$, and $\widehat{\sigma}$ with $\widehat{\sigma}\sigma_1$ in the above argument as often as necessary. Since the cardinality of the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$ decreases with each repetition, the process is guaranteed to terminate, which establishes the result. ■

Following reference [5], we construct a supervisory policy $\widehat{\mathcal{P}} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ from the supervisory policy $\mathcal{P}$ for $N(\mathbf{m}^0)$ as follows –

1) $\forall t \in T, \widehat{\mathcal{P}}(\widehat{\mathbf{m}}^0, t) = \mathcal{P}(\mathbf{m}^0, t)$.
2) Suppose $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma} \rightarrow \widehat{\mathbf{m}}^j$ in N under the supervision of $\widehat{\mathcal{P}}$,
   a) $\forall t_i \in T_u, \widehat{\mathcal{P}}(\widehat{\mathbf{m}}^j, t_i) = 1$.
   b) $\forall t_i \in T_c, (\widehat{\mathcal{P}}(\widehat{\mathbf{m}}^j, t_i) = 1) \Leftrightarrow$
      i) $t_i \in T_e(N, \widehat{\mathbf{m}}^j)$, and
      ii) $\exists \sigma \in T^*$, such that
         A) $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^k$ under the supervision of $\mathcal{P}$ in N,
         B) $\forall k \in \{1, 2, \ldots, m\}, \mathbf{x}(\widehat{\sigma} t_i)_k \geq \mathbf{x}(\sigma)_k$, and
         C) $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma} t_i)_j > \mathbf{x}(\sigma)_j\} \subseteq T_u$.

The proof of lemma 5.5 of reference [5] serves as a proof of the following result, which is skipped for brevity.

Lemma 3.4: Suppose $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ enforces liveness in $N(\mathbf{m}^0)$, where $N \in \mathcal{F}$, and $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$, then the supervisory policy $\widehat{\mathcal{P}} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ defined above, enforces liveness in $N(\widehat{\mathbf{m}}^0)$.

The following result is a direct consequence of lemma 3.4.

Theorem 3.5: Let $N \in \mathcal{F}$, where $N = (\Pi, T, \Phi, \Gamma)$ is a general FCPN structure, then the set $\Delta(N)$ is right-closed.

We turn our attention to the development of a positive test for membership in $\Delta(N)$ when $N \in \mathcal{F}$. The following result notes that testing the control-invariance of a right-closed set of markings $\mathcal{M}$ with respect to a PN structure $N = (\Pi, T, \Phi, \Gamma)$, where $T = T_u \cup T_c (T_u \cap T_c = \emptyset)$, is decidable. The proof of lemma 5.10 of reference [5] serves as a proof of the following result, and is skipped for brevity.

Lemma 3.6: The control-invariance of a right-closed set of markings $\mathcal{M} \subseteq \mathcal{N}^{card(\Pi)}$ with respect to a PN structure $N = (\Pi, T, \Phi, \Gamma)$ is decidable.

The decision procedure for control-invariance of $\mathcal{M}$ with respect to N that is in the proof of lemma 5.10 of reference [5] requires testing the following condition

$$\forall t_u \in T_u, \forall \mathbf{m}^i \in \min\{\mathcal{M}\}, \exists \mathbf{m}^j \in \min\{\mathcal{M}\}, \text{ such that } \left(\max\left\{\mathbf{IN}_u, \mathbf{m}^i\right\} + \mathbf{C} \times \mathbf{1}_u\right) \geq \mathbf{m}^j, \tag{5}$$

where $\mathbf{IN}_u$ is the u-th column of the input matrix (cf. equation 1) and $\mathbf{1}_u$ is the unit-vector where the u-th entry is unity. Given a right-closed set of markings $\mathcal{M} \subseteq \mathcal{N}^{card(\Pi)}$ that is control-invariant with respect to $N = (\Pi, T, \Phi, \Gamma)$, we define a marking monotone policy $\mathcal{P}_{\mathcal{M}} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$, as described below

$$\left(\mathcal{P}_{\mathcal{M}}(\mathbf{m}^i, t_j) = 0\right) \Leftrightarrow \left(t_j \in T_e(N, \mathbf{m}^k), \text{ where } \mathbf{m}^k = \max\{\mathbf{IN}_j, \mathbf{m}^i\}, \mathbf{m}^k \rightarrow t_j \rightarrow \mathbf{m}^l \text{ in } N, \text{ and } \mathbf{m}^l \notin \mathcal{M}\right) \tag{6}$$

The control-invariance of $\mathcal{M}$ ensures the fact that uncontrollable transitions are never disabled at any marking. For cases where $\Delta(N)$ is right-closed for some $N = (\Pi, T, \Phi, \Gamma)$, the supervisory policy $\mathcal{P}_{\Delta(N)}$ (cf. equation 6) is the minimally restrictive, marking monotone, supervisory policy that enforces liveness in $N(\mathbf{m}^0)$ for any $\mathbf{m}^0 \in \Delta(N)$.

The following theorem is about a necessary and sufficient condition for the existence of a liveness enforcing supervisory policy in general PNs, which will find use in the development of a positive test for membership in $\Delta(N)$ in lemma 3.8. The proof of theorem 5.1 in reference [6] serves as a proof this theorem, and we refrain from repeating it in the interest of space.

Theorem 3.7: Let $N(\mathbf{m}^0)$ be a general PN, where $N = (\Pi, T, \Phi, \Gamma)$. There is a liveness enforcing supervisory policy $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ if and only if $\exists \mathcal{M} \subseteq \Re(N, \mathbf{m}^0)$ that is control-invariant with respect to N, such that

1) $\forall \mathbf{m}^1 \in \mathcal{M}, \exists \mathbf{m}^2, \mathbf{m}^3 \in \mathcal{M}, \exists \sigma_1, \sigma_2 \in T^*$, such that $\mathbf{m}^1 \rightarrow \sigma_1 \rightarrow \mathbf{m}^1 \rightarrow \sigma_2 \rightarrow \mathbf{m}^3$, and
   a) $\mathbf{m}^3 \geq \mathbf{m}^2$;
   b) all transitions in T appear at least once in σ_2;
   c) $\forall \sigma_3 \in pr(\sigma_1\sigma_2), \mathbf{m}^1 \rightarrow \sigma_3 \rightarrow \mathbf{m}^4 \Rightarrow \mathbf{m}^4 \in \mathcal{M}$, where $pr(\bullet)$ is the prefix set of the string argument; and
2) $\mathbf{m}^0 \in \mathcal{M}$.

Lemma 3.8: Suppose $\mathbf{m}^0 \in \Delta(N)$ for some $N \in \mathcal{F}$ where $N = (\Pi, T, \Phi, \Gamma)$, then there is a closed-path $v \rightarrow \sigma \rightarrow v$ in the coverability graph $G(N, \mathbf{m}^0, \mathcal{P}_{\Delta(N)})$ (cf. equation 6), such that

1) all transitions appear at least once in σ (i.e. $\mathbf{x}(\sigma) \geq \mathbf{1}$), and
2) $\mathbf{Cx}(\sigma) \geq \mathbf{0}$.

Proof: Since $\mathcal{P}_{\Delta(N)}$ enforces liveness in $N(\mathbf{m}^0)$, from theorem 3.7 we have $\mathbf{m}^0 \rightarrow \sigma_1 \rightarrow \mathbf{m}^1 \rightarrow \sigma_2 \rightarrow \mathbf{m}^3$ under $\mathcal{P}_{\Delta(N)}$ in N. Additionally, $\mathbf{x}(\sigma_2) \geq \mathbf{1}$ and $\mathbf{Cx}(\sigma_2) \geq \mathbf{0}$. This would mean that there must be a path in the coverability graph $G(N, \mathbf{m}^0, \mathcal{P}_{\Delta(N)})$ that corresponds to the firing string $\sigma_1\sigma_2\sigma_2$. Specifically, this path must be of the form $v_0 \rightarrow \sigma_1\sigma_2 \rightarrow v_1 \rightarrow \sigma_2 \rightarrow v_1$. Therefore, there is a closed-path with the required properties in the coverability graph as stipulated. ■

The following result is about a necessary and sufficient condition for the existence of a liveness enforcing supervisory policy in $N(\mathbf{m}^0)$ where $N = (\Pi, T, \Phi, \Gamma)$ and $N \in \mathcal{F}$.

Lemma 3.9: There is a liveness enforcing supervisory policy for $N(\mathbf{m}^0)$ where $N = (\Pi, T, \Phi, \Gamma)$ and $N \in \mathcal{F}$, if and only if $\exists \mathcal{M} \subseteq \mathcal{N}^{card(\Pi)}$ that is right-closed and control-invariant with respect to N, and $\forall \mathbf{m} \in \min\{\mathcal{M}\}$, there is a closed-path $v \rightarrow \sigma \rightarrow v$ in the coverability graph $G(N, \mathbf{m}, \mathcal{P}_{\mathcal{M}})$ such that

1) all transitions appear at least once in σ (i.e. $\mathbf{x}(\sigma) \geq \mathbf{1}$), and
2) $\mathbf{Cx}(\sigma) \geq \mathbf{0}$.

Proof: (*Only if*) If there is a liveness enforcing supervisory policy for $N(\mathbf{m}^0)$, then $\mathbf{m}^0 \in \Delta(N)$. We let $\mathcal{M} = \Delta(N)$ be the right-closed and control-invariant set. Letting $\mathbf{m}^0$ to be each minimal element of $\Delta(N)$, we get the stipulated condition on the coverability graph from lemma 3.8.

(*If*) Suppose there is a right-closed, control-invariant set $\mathcal{M}$ where the stipulated condition is true in the coverability graph $G(N, \mathbf{m}^i, \mathcal{P}_{\mathcal{M}})$ for every $\mathbf{m}^i \in \min\{\mathcal{M}\}$, then the same must be true for any $\mathbf{m}^0 \in \mathcal{M}$. That is, there is a closed-path $v \rightarrow \sigma \rightarrow v$ in $G(N, \mathbf{m}^0, \mathcal{P}_{\mathcal{M}})$. Since the vertex v is connected to the root-vertex of the coverability graph, $\exists \widehat{\sigma} \in T^*$ such that $\mathbf{m}^0 \rightarrow \widehat{\sigma} \rightarrow \mathbf{m}^1 \rightarrow \widehat{\sigma} \rightarrow \mathbf{m}^2$ under the supervision of $\mathcal{P}_{\mathcal{M}}$ in N, $\mathbf{x}(\widehat{\sigma}) \geq \mathbf{1}$, and $\mathbf{m}^2 \geq \mathbf{m}^1$. Therefore, $\mathcal{P}_{\mathcal{M}}$ enforces liveness in $N(\mathbf{m}^0)$. ■

From lemma 3.9, we get the following result which says there is a positive-test for membership in $\Delta(N)$ if $N \in \mathcal{F}$. The proof of lemma 5.14 of [5] also serves as a proof of this result, and is therefore skipped for brevity. Essentially, the proof notes that there is a liveness enforcing supervisory policy for $N(\mathbf{m}^0)$ (i.e. $\mathbf{m}^0 \in \Delta(N)$) if and only if there is a control-invariant, right-closed set of markings $\mathcal{M}$ where each minimal element of $\mathcal{M}$ meets the conditions stipulated in lemma 3.9. The positive-test for membership in $\Delta(N)$ proceeds via an exhaustive search of the set of minimal elements of control-invariant, right-closed set of markings.

Lemma 3.10: Suppose $N = (\Pi, T, \Phi, \Gamma)$ and $N \in \mathcal{F}$, then there is a positive test for membership in $\Delta(N)$.

As an illustration, consider the general FCPN $N_3 \in \mathcal{F}$ shown in figure 2(c). We seek a positive test for $\mathbf{m}_3^0 \in \Delta(N_3)$ where $\Delta(N_3) \neq \emptyset$. This test proceeds by a brute-force enumeration of all possible minimal elements for $\Delta(N_3)$, which is followed by a test for the conditions of lemma 3.9. At some point of this enumerative search, the true minimal elements of $\Delta(N_3)$ will be found, followed by a positive result of membership of $\mathbf{m}_3^0 \in \Delta(N_3)$. Suppose the exhaustive search for minimal elements resulted in the set $\{(2\ 0)^T, (0\ 2)^T\}$. The right-closed set with these minimal elements is control-invariant (cf equation 5). As pointed out in section II-A, the coverability graph $G(N_3, (2\ 0)^T, \mathcal{P}_{\mathcal{M}_3})$, where $\min\{\mathcal{M}_3\} = \{(2\ 0)^T, (0\ 2)^T\}$ has a closed-path $(\omega\ \omega) \rightarrow t_2t_1t_2t_1t_3 \rightarrow (\omega\ \omega)$ with the stipulated conditions of lemma 3.9. That is, all transitions appear at least once in the string $t_2t_1t_2t_1t_3$ and $\mathbf{C}_2\mathbf{x}(t_2t_1t_2t_1t_3) = (1\ 0)^T$. Similarly, there is a closed path $(\omega\ \omega) \rightarrow\rightarrow t_2t_1t_2t_1t_3 \rightarrow (\omega\ \omega)$ in $G(N_3, (0\ 2)^T, \mathcal{P}_{\mathcal{M}_3})$, which meets the conditions of lemma 3.9. At this point, we can say there is a supervisory policy that enforces liveness in $N_3(\mathbf{m}_3^0)$ for any $\mathbf{m}_3^0$ that belongs to the control-invariant, right-closed set whose minimal elements are $\{(2\ 0)^T, (0\ 2)^T\}$.

The right-closed set $\mathcal{M}_4$ whose minimal elements are

$$\min\{\mathcal{M}_4\} = \{(0\ 0\ 1\ 0)^T, (2\ 0\ 0\ 0)^T, (1\ 0\ 0\ 1)^T, (0\ 2\ 0\ 0)^T, (0\ 0\ 0\ 2)^T\}.$$

is control-invariant with respect to the general FCPN structure N_4 shown in figure 1(d) (cf. equation 5). Figure 4 shows the coverability graphs $G(N_4, \mathbf{m}^i, \mathcal{P}_{\mathcal{M}_4})$ for each $\mathbf{m}^i \in \min\{\mathcal{M}_4\}$. In each of these coverability graphs there is a closed-path $(\omega\ \omega\ \omega\ \omega) \rightarrow t_4t_2t_1t_5t_6t_3 \rightarrow (\omega\ \omega\ \omega\ \omega)$, where $\mathbf{C}_4\mathbf{x}(t_4t_2t_1t_5t_6t_3) = \mathbf{0}$. Since $N_4 \in \mathcal{F}$, from lemma 3.9 we infer that there is a liveness enforcing supervisory policy, $\mathcal{P}_{\mathcal{M}_4}$, that enforces liveness in the general FPCN $N_4(\mathbf{m}_4^0)$ for any $\mathbf{m}_4^0 \in \mathcal{M}_4$.

The right-closed set $\mathcal{M}_5$ whose minimal elements are $\{(1\ 0\ 0\ 0\ 1)^T, (0\ 0\ 0\ 1\ 1)^T\}$ is control-invariant with respect the general FCPN structure N_5 shown in figure 1(e). The coverability graphs $G(N_5, (1\ 0\ 0\ 0\ 0)^T, \mathcal{P}_{\mathcal{M}_5})$ and $G(N_5, (0\ 0\ 0\ 1\ 1)^T, \mathcal{P}_{\mathcal{M}_5})$ are shown in figure 5(a) and 5(b) respectively. In each of these coverability graphs we see the closed path $(\omega\ \omega\ \omega\ \omega\ \omega) \rightarrow t_7t_2t_3t_6t_2t_4t_5t_4t_5t_7t_2t_4t_5t_1 \rightarrow (\omega\ \omega\ \omega\ \omega\ \omega)$ and $\mathbf{C}_5\mathbf{x}(t_7t_2t_3t_6t_2t_4t_5t_4t_5t_7t_2t_4t_5t_1) = (0\ 2\ 0\ 1\ 1)^T$. Therefore, there is a supervisory policy (i.e. $\mathcal{P}_{\mathcal{M}_5}$) that enforces liveness in $N_5(\mathbf{m}_5^0)$ for any $\mathbf{m}_5^0 \in \mathcal{M}_5$. This observation is inconclusive if we are interested in knowing if there is a supervisory policy that enforces liveness in $N_5(\widehat{\mathbf{m}}_5^0)$ where $\widehat{\mathbf{m}}_5^0 = (0\ 1\ 1\ 0\ 0)^T$, as $\widehat{\mathbf{m}}_5^0 \notin \mathcal{M}_5$. After all, there might be another control-invariant, right-closed set that contains $\widehat{\mathbf{m}}_5^0$ whose minimal elements meet the requirement of lemma 3.9, it could very well be that we have not yet found it.

In order to certify that there is no liveness enforcing policy for $N_5(\widehat{\mathbf{m}}_5^0)$, we need a positive test for the non-existence of a supervisory policy that enforces liveness in an arbitrary general FCPN from the class $\mathcal{F}$. This issue is addressed in the remainder of this section.

The following theorem is about a necessary and sufficient condition for the existence of a supervisory policy that enforces liveness in general PNs where all transitions are controllable. The discussion following the proof of theorem 5.1 in reference [5], which is originally stated for ordinary PNs, serves as a proof of the following theorem, and is skipped for brevity.

Theorem 3.11: Suppose $N = (\Pi, T, \Phi, \Gamma)$ is a general PN structure where all transitions are controllable (i.e. $T = T_c$, or $T_u = \emptyset$). There is a liveness enforcing supervisory policy for the general PN $N(\mathbf{m}^0)$ if and only if $\exists \sigma_1, \sigma_2 \in T^*$ such that

1) $\mathbf{m}^0 \rightarrow \sigma_1 \rightarrow \mathbf{m}^1 \rightarrow \sigma_2 \rightarrow \mathbf{m}^2$ in N,
2) all transitions appear at least once in σ_2 (i.e. $\mathbf{x}(\sigma_2) \geq \mathbf{1}$, and
3) $\mathbf{m}^2 \geq \mathbf{m}^1$ (i.e. $\mathbf{Cx}(\sigma_2) \geq \mathbf{0}$),

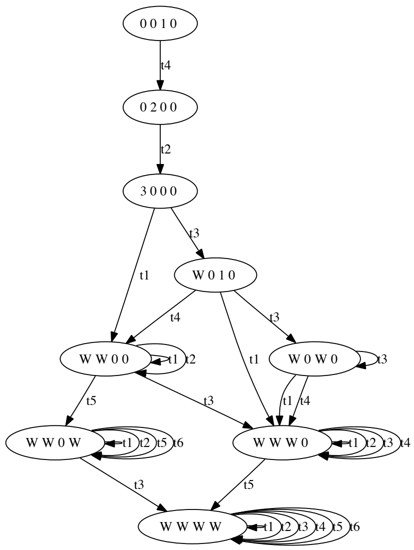
where $\mathbf{C}$ is the incidence matrix of N, and $\mathbf{x}(\bullet)$ is the *score* or *Parkih-mapping* of the string argument.

Let $N_f = (\Pi, T, \Phi, \Gamma)$ be the structure that results when we assume all transitions in a general PN structure $N = (\Pi, T, \Phi, \Gamma), (T = T_u \cup T_c; T_u \cap T_c = \emptyset)$ are controllable. Following reference [5] we define the set

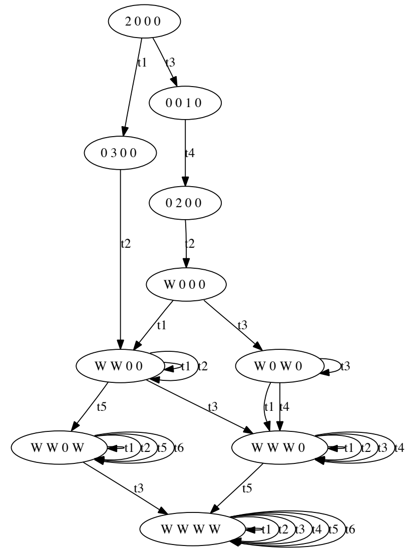
$$\Delta_f(N) = \{\mathbf{m}^0 \in \mathcal{N}^{card(\Pi)} \mid \exists \mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\} \text{ that enforces liveness in } N_f(\mathbf{m}^0)\} \quad (7)$$

We note that $\Delta_f(N)$ is right-closed, as a consequence of theorem 3.11 (cf. proof of observation 4.1, [5]). For example, $\Delta_f(N_5)$ for the general FCPN structure N_5 shown in figure 1(e) is defined by the minimal elements

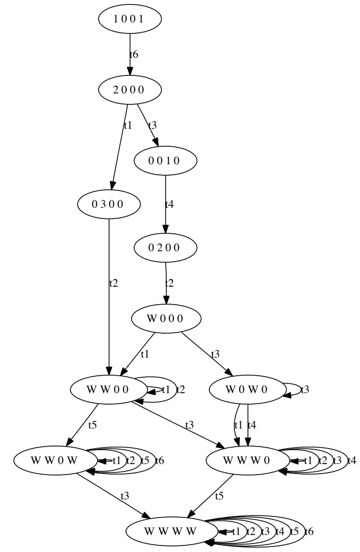
$$\{(1\ 0\ 0\ 0\ 0)^T, (0\ 2\ 0\ 0\ 0)^T, (0\ 1\ 1\ 0\ 0)^T, (0\ 1\ 0\ 1\ 0)^T, (0\ 1\ 0\ 0\ 1)^T, (0\ 0\ 1\ 1\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}.$$



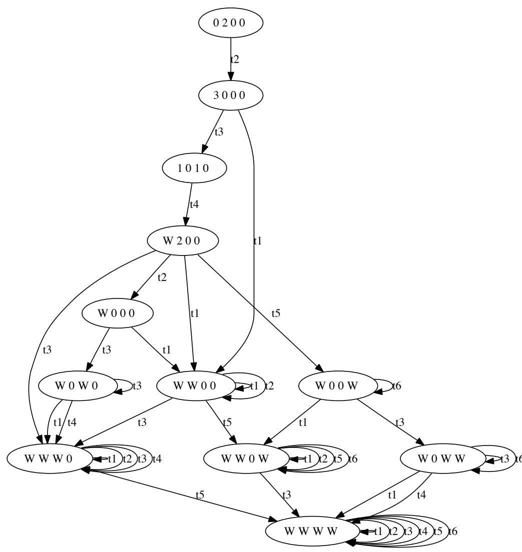
(a) $G(N_4, \mathbf{m}^1, \mathcal{P}_{\mathcal{M}})$



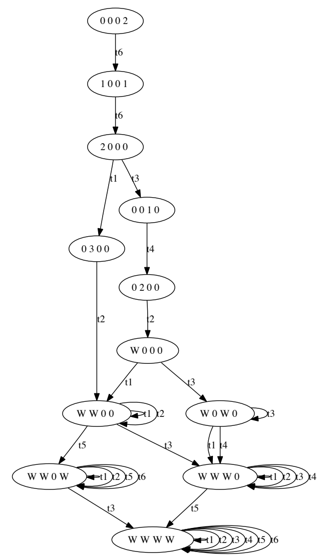
(b) $G(N_4, \mathbf{m}^2, \mathcal{P}_{\mathcal{M}})$



(c) $G(N_4, \mathbf{m}^3, \mathcal{P}_{\mathcal{M}})$



(d) $G(N_4, \mathbf{m}^4, \mathcal{P}_{\mathcal{M}})$



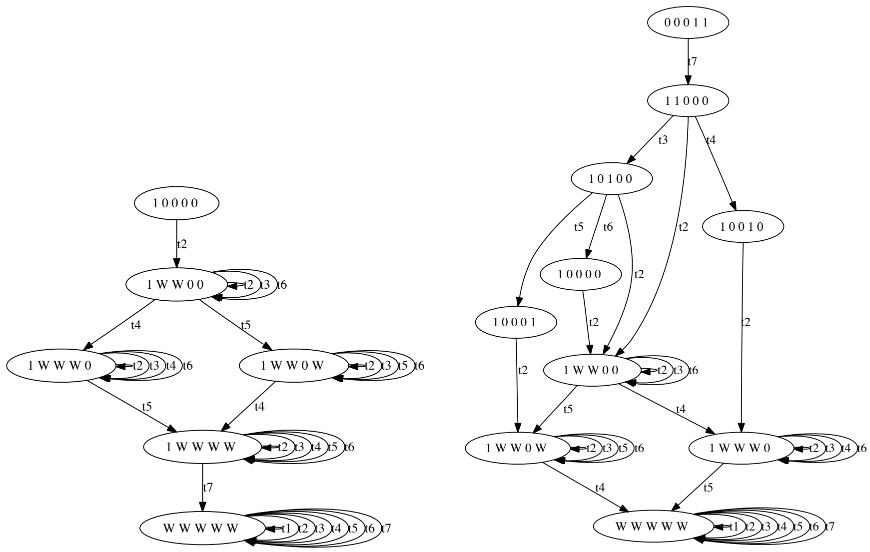
(e) $G(N_4, \mathbf{m}^5, \mathcal{P}_{\mathcal{M}})$

Fig. 4. (a) $G(N_4, \mathbf{m}^1, \mathcal{P}_{\mathcal{M}})$ where $\mathbf{m}^1 = (0\ 0\ 1\ 0)^T$, (b) $G(N_4, \mathbf{m}^2, \mathcal{P}_{\mathcal{M}})$ where $\mathbf{m}^2 = (2\ 0\ 0\ 0)^T$, (c) $G(N_4, \mathbf{m}^3, \mathcal{P}_{\mathcal{M}})$ where $\mathbf{m}^3 = (1\ 0\ 0\ 1)^T$, (d) $G(N_4, \mathbf{m}^1, \mathcal{P}_{\mathcal{M}})$ where $\mathbf{m}^4 = (0\ 2\ 0\ 0)^T$, and (e) $G(N_4, \mathbf{m}^5, \mathcal{P}_{\mathcal{M}})$ where $\mathbf{m}^5 = (0\ 0\ 0\ 2)^T$.

The procedure listed in figure 6, which is essentially the procedure in figure 8 of reference [5], is a positive-test for the non-existence of a liveness enforcing supervisory policy for a general FCPN $N(\mathbf{m}^0)$ where $N = (\Pi, T, \Phi, \Gamma)$ and $N \in \mathcal{F}$. The right-closed set $\Delta_i(N)$ serves as an outer-approximant of $\Delta(N)$ at each stage of execution of this procedure. The procedure starts with $\Delta_0(N) = \Delta_f(N)$ and proceeds to replace $\Delta_i(N)$ with a smaller set $\Delta_{i+1}(N)$ (i.e. $\Delta_i(N) \supset \Delta_{i+1}(N)$) whenever one of two conditions are violated:

1) $\Delta_i(N)$ is not control-invariant with respect to N (cf. lines 3 to 6), or
2) The coverability graph $G(N, \mathbf{m}, \mathcal{P}_{\Delta_i(N)})$ does not have the property identified in lemma 3.9 for some $\mathbf{m} \in \min\{\Delta_i(N)\}$ (cf. lines 10 to 19).

In line 4, when $\Delta_i(N)$ is not control-invariant with respect to N, each minimal element of $\Delta_i(N)$ is elevated by the smallest vector that will yield a set of vectors that meet the controllability requirement (cf. equation 8). The set of elevated vectors need not be minimal, therefor in line 5, the minimal vectors that represent



(a) $G(N_5, (1\ 0\ 0\ 0\ 0)^T, \mathcal{P}_{\mathcal{M}_5})$ (b) $G(N_5, (10\ 0\ 0\ 1\ 1)^T, \mathcal{P}_{\mathcal{M}_5})$

Fig. 5. (a) $G(N_5, (1\ 0\ 0\ 0\ 0)^T, \mathcal{P}_{\mathcal{M}_5})$ and (b) $G(N_5, (0\ 0\ 0\ 1\ 1)^T, \mathcal{P}_{\mathcal{M}_5})$, $\min\{\mathcal{M}_5\} = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}$.

the elevated set of vectors are used to define the next iterate of $\Delta_i(N)$.

Following this, the procedure checks to see if the coverability graphs of each minimal element of a right-closed, control-invariant set $\Delta_i(N)$ meets the closed-path condition of lemma 3.9. If there is a minimal element of $\Delta_i(N)$ for which this condition is not met, then the minimal element is elevated by a unit-vector $\mathbf{1}_j \in \{0, 1\}^{card(\Pi)}$ only if the placement of an unbounded number (i.e. ω-many tokens) in the j-th place will yield a coverability graph with the closed-path property of lemma 3.9.

There is no supervisory policy that enforces liveness in $N(\mathbf{m}^0)$ if $\mathbf{m}^0 \notin \Delta_i(N)$ at some point of execution of the procedure in figure 6. This claim is established by an induction argument over the index i. The base case is established for $i = 0$ as there can be no liveness enforcing supervisory policy if $\mathbf{m}^0 \notin \Delta_0(N)(= \Delta_f(N))$. The induction hypothesis supposes there is no liveness enforcing supervisory policy if $\mathbf{m}^0 \notin \Delta_i(N)$ for some i, and the induction step ponders the case when $\mathbf{m}^0 \notin \Delta_{i+1}(N)$. If this conclusion was reached at the end of line 6, then there is an uncontrollable transition that can be fired at $\mathbf{m}^0$ that will result in a marking that is not in $\Delta_i(N)$, and by the induction hypothesis there is no liveness enforcing policy for this new marking. This leads us to the conclusion that such a policy does not exist for $\mathbf{m}^0$ either. On the other hand, if the conclusion $\mathbf{m}^0 \notin \Delta_{i+1}(N)$ was reached at the end of line 19, then it means that any attempt at keeping the markings in $\mathfrak{R}(N, \mathbf{m}^0, \mathcal{P}_{\Delta_i(N)})$ to remain within $\Delta_i(N)$ will essentially result in a supervisory policy induced loss of liveness. Therefore, there is no supervisory policy that enforces liveness in $N(\mathbf{m}^0)$, which completes the induction argument. So, if $\mathbf{m}^0 \notin \Delta(N)$, there will be an index i where this will be realized, which in turns results in the following observation.

Lemma 3.12: There is positive test for the non-existence of a liveness enforcing supervisory policy for a general FCPN $N(\mathbf{m}^0)$, where $N = (\Pi, T, \Phi, \Gamma)$ and $N \in \mathcal{F}$.

As an illustration, the procedure shown in figure 6 when executed for $N_5(\mathbf{m}^0)$ where $\mathbf{m}^0 = (0\ 1\ 1\ 0\ 0)^T$, would start with the following elements of the set $\min\{\Delta_0(N_5)\}$

$$\{(1\ 0\ 0\ 0\ 0)^T, (0\ 2\ 0\ 0\ 0)^T, (0\ 1\ 1\ 0\ 0)^T, (0\ 1\ 0\ 1\ 0)^T, (0\ 1\ 0\ 0\ 1)^T, (0\ 0\ 1\ 1\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}.$$

At the end of line 6, we would have

$$\min\{\Delta_1(N_5)\} = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}.$$

Since $\mathbf{m}^0 = (0\ 1\ 1\ 0\ 0)^T \notin \Delta_1(N)$, we conclude that there is no supervisory policy that enforces liveness in $N_5(\mathbf{m}^0)$.

positive-test-for-non-existence ((general FCPN) $N(\mathbf{m}^0)$ where $N \in \mathcal{F}$)

1: $i = 0$. {/* Iteration Count for $\Delta_i(N)$ */}
2: $\Delta_i = \Delta_f(N)$ (cf. equation 7), and let $\{\widetilde{\mathbf{m}}^i\}_{i=1}^k = \min\{\Delta_i(N)\}$.
3: **while** $\mathbf{m}^0 \in \Delta_i(N)$ **do**
4: **while** ($\mathbf{m}^0 \in \Delta_i(N), \exists t_u \in T_u$, such that $(\max\{\mathbf{IN}_u, \widetilde{\mathbf{m}}^i\} + \mathbf{C} \times \mathbf{1}_u) \notin \Delta_i(N)$ (i.e. equation 5 is violated for $\widetilde{\mathbf{m}}^i$ and $t_u \in T_u$)) **do**
5: Replace $\widetilde{\mathbf{m}}^i$ by a set of $k-1$ vectors $\{\widehat{\mathbf{m}}^l\}_{l=1}^{k-1}$($i.e.\widetilde{\mathbf{m}}^i \leftarrow \{\widehat{\mathbf{m}}^l\}_{l=1}^{k-1}$) where for each $j \in \{1, 2, \ldots, k\} - \{i\}$, create a new marking $\widehat{\mathbf{m}}^l$, given by the expression
$$\widehat{\mathbf{m}}^l = \widetilde{\mathbf{m}}^i + \max\left\{\mathbf{0}, \widetilde{\mathbf{m}}^j - \left(\max\{\mathbf{IN} \times \mathbf{1}_u, \widetilde{\mathbf{m}}^i\} + \mathbf{C} \times \mathbf{1}_u\right)\right\}, \tag{8}$$
6: Replace the resulting set of $\{\widetilde{\mathbf{m}}^i\}_i$ vectors by their minimal elements, and modify the value of k to equal the size of the minimal set of vectors; $i \leftarrow i+1$; and $\Delta_i(N)$ is the right-closed set identified by this minimal set of vectors.
7: **end while**{ /* Either ($\Delta_i(N)$ is control-invariant w.r.t. N and $\mathbf{m}^0 \in \Delta_i(N)$) or ($\mathbf{m}^0 \notin \Delta_i(N)$) upon exit */}
8: **if** $\forall \widetilde{\mathbf{m}}^i \in min(\Delta_i(N)), G(N(\widetilde{\mathbf{m}}^i), \widetilde{\mathcal{P}}_{\Delta_i(N)})$ has the path identified in lemma 3.9 **then**
9: Hang indefinitely (or, **return** ("*there is a solution*"))
10: **else**
11: **for** $\widetilde{\mathbf{m}}^i$ where $G(N(\widetilde{\mathbf{m}}^i), \widetilde{\mathcal{P}}_{\Delta_i(N)})$ does not have the path identified in lemma 3.9 **do**
12: Replace $\widetilde{\mathbf{m}}^i$ by the set
$$\{\widetilde{\mathbf{m}}^i + \mathbf{1}_j \mid j \in \{1, 2, \ldots, n\}\} \tag{9}$$
13: **end for**
14: Replace the resulting set of $\{\widetilde{\mathbf{m}}^i\}_i$ vectors by their minimal elements, and modify the value of k to equal the size of the minimal set of vectors; $i \leftarrow i+1$; $\Delta_i(N)$ is the right-closed set identified by this minimal set of vectors.
15: **if** $\mathbf{m}^0 \notin \Delta_i(N)$ **then**
16: **return** ("*no solution*")
17: **end if**
18: **end if**
19: **end while**{ /* $\mathbf{m}^0 \notin \Delta_i(N)$ upon exit */}
20: **return** ("*no solution*")

Fig. 6. A positive-test for the non-existence of a supervisory policy that enforces liveness in an arbitrary general FCPN $N(\mathbf{m}^0)$ where $N = (\Pi, T, \Phi)$ (cf. Figure 8, [5]).

When the procedure in figure 6 is executed with input $N_1(\mathbf{m}_1^0)$, where $\mathbf{m}_1^0 = (0\ 1\ 0\ 0\ 0\ 0\ 0)^T$, it would start with $\Delta_0(N_1) = \Delta_f(N_1)$, where
$$\min\{\Delta_0(N_1)\} = \{(1\ 0\ 0\ 0\ 0\ 0\ 0)^T, (0\ 1\ 0\ 0\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0\ 0\ 0)^T\}.$$
Following line 6 of this procedure it yield the first iterate $\Delta_1(N_1)$, where
$$\min\{\Delta_1(N_1)\} = \{(1\ 0\ 0\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0\ 0\ 0)^T\}.$$
Since $\mathbf{m}_1^0 \notin \Delta_1(N_1)$, the procedure terminates with the conclusion that $\mathbf{m}_1^0 \notin \Delta(N_1)$, which in turn implies that there is no supervisory policy that enforces liveness in $N_1(\mathbf{m}_1^0)$.

Lemmas 3.10 and 3.12 together imply the following result.

Theorem 3.13: The existence of a liveness enforcing supervisory policy for $N(\mathbf{m}^0)$ is decidable if $N = (\Pi, T, \Phi, \Gamma)$ is a general PN such that $N \in \mathcal{F}$.

IV. Concluding Remarks

Theorem 3.2 identifies the minimally restrictive supervisory policy that enforces liveness in an arbitrary PN structure that is not necessarily from the class $\mathcal{F}$. Since $\Delta(N)$ is control-invariant for any PN structure N (cf. lemma 3.1), the firing of any uncontrollable transition from a marking in $\Delta(N)$ will always result in a new marking that is $\Delta(N)$. Stated differently, only the firing of a controllable transition at a marking in $\Delta(N)$ could possibly result in a marking that is not in $\Delta(N)$. Therefore, a supervisory policy that control disables a controllable transition at a marking whenever its firing results in a marking that is not in $\Delta(N)$ would enforce liveness in $N(\mathbf{m}^0)$ whenever $\mathbf{m}^0 \in \Delta(N)$. These observations are not restricted to any specific class of PN structures.

If we restrict attention to the class of general FCPN structures $\mathcal{F}$ identified in this paper, which strictly includes the class of ordinary FCPN structures, we have the additional observation that the set $\Delta(N)$ is right-closed for any $N \in \mathcal{F}$. This, along with the observations of the previous paragraph, implies that if there is a supervisory policy that enforces liveness in $N(\mathbf{m}^0)$ for some $\mathbf{m}^0$, then without loss of generality we can assume there is a marking-monotone liveness enforcing policy for $N(\mathbf{m}^0)$. If a transition is control-enabled at a marking under a marking monotone policy, it remains control-enabled at any marking that is larger than the original marking. This permits the extension of the standard procedure for generating coverability graphs of a PN to include the influence of a marking monotone supervisory policy that disables a controllable transition whenever its firing would result in a new marking that is not in some right-closed set of markings. Theorem 3.9 notes that there is a liveness enforcing supervisory policy in a PN instance $N(\mathbf{m}^0)$ that belongs to a family where the set $\Delta(N)$ is right-closed if and only if there is a closed-path in the above mentioned variant of the coverability graph which satisfies a specific property. The positive-test for the existence of a liveness enforcing supervisory policy follows from a brute-force enumeration of right-closed, control-invariant sets followed by a test for the closed-path requirement of theorem 3.9. The positive-test for the non-existence of a supervisory policy that enforces liveness uses a sequence of iterates $\{\Delta_i(N)\}_{i=0}$, where $\forall i, (\Delta(N) \subseteq)\Delta_i(N) \subset \Delta_{i-1}(N)$. The specifics of the iterative procedure guarantees that if there is no supervisory policy that enforces liveness in $N(\mathbf{m}^0)$ then there is an index i such that $\mathbf{m}^0 \notin \Delta_i(N)$.

It is important to note that these observations apply equally to any family of PN structures where the set $\Delta(N)$ is right-closed for any instance N in the class. There are general FCPN structures N where $N \notin \mathcal{F}$, and yet $\Delta(N)$ is right-closed (cf. the general FCPN structure N_8 shown in figure 1(f)). We suggest explorations into extending the class $\mathcal{F}$ to a larger family that posses the right-closure property as a direction for future research.

The procedures in this paper and reference [5] are not applicable to those instances where the above mentioned desideratum of right-closure is not satisfied. For instance, a liveness enforcing supervisory policy will have to permit transition t_2 at the marking $(0\ 1\ 0\ 0\ 0)^T$ in the general FCPN structures N_7 (N_8) shown in figures 3(a) (3(b)) – otherwise there would be a policy-induced deadlock at this marking. It is also mandatory that such a policy disable transition t_2 at the marking $(0\ 1\ 0\ 0\ 1)^T (\geq (0\ 1\ 0\ 0\ 0)^T)$. That is, any policy that enforces liveness in $N_7(\mathbf{m}_7^0)$ ($N_8(\mathbf{m}_8^0)$) for an appropriate choice of $\mathbf{m}_7^0$ ($\mathbf{m}_8^0$) are essentially *not* marking monotone. We suggest explorations into decision procedures for the existence of liveness enforcing supervisory policy in $N(\mathbf{m}^0)$ where N is a general FCPN structure where $\Delta(N)$ is not right-closed, as another future research direction. This will essentially require the identification of an alternate, finite-characterization of the set $\Delta(N)$. This characterization was accomplished by the finite set of minimal elements of $\Delta(N)$ when it was right-closed. The decision algorithms utilized these minimal elements to identify situations where there was, or there was not, a liveness enforcing supervisory policy for some $N(\mathbf{m}^0)$ for the cases where $\Delta(N)$ was right-closed. We surmise that something similar is necessary with the finite characterization of the set $\Delta(N)$, for those cases where it is not right-closed.

References

[1] J. Peterson, *Petri Net Theory and the Modeling of Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1981.

[2] M. Hack, “Analysis of production schemata by Petri nets,” Master’s thesis, Massachusetts Institute of Technology, February 1972.

[3] J. Desel and J. Esparza, *Free Choice Petri Nets*. Cambridge, UK: Cambridge Tracts in Theoretical Computer Science, 1995.

[4] T. Murata, “Petri nets: Properties, analysis and applications,” *Proceedings of the IEEE*, vol. 77, no. 4, pp. 541–580, 1989.

[5] R. Sreenivas, “On the Existence of Supervisory Policies that Enforce Liveness in Partially Controlled Free-Choice Petri Nets,” *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 435–449, February 2012.

[6] ——, “On the existence of supervisory policies that enforce liveness in discrete-event dynamic systems modeled by controlled Petri nets,” *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 928–945, July 1997.

[7] M. V. Iordache and P. J. Antsaklis, “Design of $\mathcal{T}$-Liveness Enforcing Supervisors in Petri Nets,” *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 1202–1218, November 2003.

[8] R. Valk and M. Jantzen, “The residue of vector sets with applications to decidability problems in Petri nets,” *Acta Informatica*, vol. 21, pp. 643–674, 1985.

[9] P. Ramadge and W. Wonham, “Modular feedback logic for discrete event systems,” *SIAM J. Control and Optimization*, vol. 25, no. 5, pp. 1202–1218, September 1987.

[10] W. Reisig, *Petri Nets*. Berlin: Springer-Verlag, 1985.

[11] M. Garey and D. Johnson, *Computers and Intractability: A guide to the theory of NP-completeness*. New York: W.H. Freeman and Company, 1979.

[12] K. Barkaoui and J. Pradat-Peyre, “On Liveness and Controlled Siphons in Petri Nets,” vol. 1091, January 1996, Proc. the 17th International Conference on Applications and Theory of Petri Nets, Osaka, Japan, pages 57-72.

[13] A. Giua, “Petri nets as discrete event models for supervisory control,” Ph.D. dissertation, ECSE Dept., Rensselaer Polytechnic Institute, Troy, NY., 1992.

[14] J. Moody and P. Antsaklis, *Supervisory Control of Discrete Event Systems using Petri Nets*. MA: Kluwer Academic Publishers, 1998.

[15] M. Iordache and P. Ansaklis, *Supervisory control of Concurrent Systems: A Petri net Structural Approach*. MA: Kulwer Academic Publishers, 2006.

[16] S. Reveliotis, *Real-Time Management of Resource Allocation Systems: A Discrete-Event Systems Approach*. NY: Springer, 2005.

[17] S. Reveliotis, E. Roszkowska, and J. Choi, “Generalized algebraic deadlock avoidance policies for sequential Resource Allocation Systems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 2345–2350, December 2007.

[18] A. Ghaffari, N. Rezg, and X. Xie, “Design of a Live and Maximally Permissive Petri net Controller using the Theory of Regions,” *IEEE Transactions on Robotics and Automation*, vol. 19, no. 1, pp. 137–142, January 2003.