On Supervisory Policies that Enforce Liveness in Controlled Petri Nets that are Similar

E. Salimi[1], N. Somnath[1] and R.S. Sreenivas[1]

***Abstract*— In this paper we consider event-driven systems, modeled by *Petri nets* (PNs) that are prone to *livelocks* when left unsupervised. There are well-known conditions for the existence of *supervisory policies* that can regulate the sequence of task-executions in such a manner that there are no livelocks in the supervised system. We consider the problem of modifying a supervisory policy that ensures livelock-freedom for a given event-driven system into a similar policy for another event-driven system.**

The formal concept of *simulation* was introduced in the *Petri net* (PN) literature to capture the notion of "similarity" among PNs. In this paper we generalize this concept to *controlled* PNs.

The set of transitions in a controlled PN are partitioned into *controllable*- and *uncontrollable*-subsets. The *supervisory policy* determines which of a subset of *controllable* transitions are permitted to fire at any marking of the PN. A transition in a controlled PN can fire if and only if (1) there are sufficient tokens in its input places, *and* (2) the supervisory policy permits it to fire. A transition in a supervised PN is *live* if it can be fired, not necessarily immediately, from every marking that is reachable under supervision. The synthesis of a *liveness enforcing supervisory policy* (LESP) for a subset of transitions in a PN serves as the motivating problem for this work.

We consider two PNs that in a simulation relationship under supervision, and we derive a necessary and sufficient condition for the existence of a LESP for a subset of transitions in these PNs. Using an illustrative example, we show how this observation can be used to synthesize LESPs in a PN that simulates another PN. Further investigations into formalizing this synthesis procedure is suggested as a future research topic.

I. Introduction

If a system enters into a state where a task enters into a state of suspended animation for perpetuity, we say it is in a *livelocked* state. In contrast, all tasks of the system remain suspended for perpetuity in a *deadlocked* state of the system. A livelock-free system can never experience deadlocks, but the converse is not necessarily true. This paper is about livelock-prone systems modeled by *Petri nets* (PNs) , that can be made livelock-free under the influence of a *supervisory policy*, which regulates which task is to be executed at any given time in the system. More specifically, we consider the problem of modifying a supervisory policy that ensures livelock-freedom for a given system into a corresponding policy for second system that is "similar" to the first.

PNs have been widely used to model event-driven systems (cf. [1], [2] for a detailed treatment). Informally, a PN is a collection of *places*, *transitions*, *weighted-arcs*, and *tokens*. In graphical representation of PNs, places are represented by circles; transitions by rectangles; arcs by directed edges that either originates from a place (resp. transition) to a transition (resp. place). Filled circles that reside in places represent tokens. The term *marking* is used to refer to the distribution of resources in the system – that is, it denotes the number of tokens assigned to each place.

Transitions, which are the rectangles in the graphical representation, represent activities; Places, which are represented by circles, denote resources; and, the marking denotes the distribution of resources within the system. Each activity in the system can be started only when there are sufficient input resources, and these resources are consumed in course of its execution. Upon the conclusion of the activity, new resources are created that can be used by other activities in the system. In the PN model, weighted arcs represent the input resource requirement from places to transitions; and the resource generation that results from the conclusion of a task from transitions to places. In terms of the semantics of the PN this is represented as follows – when there are sufficient tokens in the input places to a transition, it can *fire*, which results in

1) the erasure of the appropriate number of tokens in the input places, followed by
2) the placement of an appropriate number of tokens in the output places of a transition.

The exact number of tokens that are consumed (resp. generated) by the process of transition firing is determined by the weights of the input arcs to the transition (resp. output arcs from the transition). The process of transition firing captures the change of resource distribution within the modeled system, which results in a new distribution of resources, and the process repeats as often as necessary. The initial resource distribution is represented by the initial token distribution. A plethora of transition firings can result in a plethora of possible resource allocations – this is the set of *reachable markings* of the PN model.

Since changes in resource distribution can only result from the execution of events, there are no arcs from places to places in a PN model. Similarly, there are no arcs from transitions to transitions in the PN model either.

A *transition* in a PN is said to be *live* if it can be fired from every reachable marking, although not necessarily immediately. The *supervisory control* of PNs supposes the existence of a set of *controllable* (*uncontrollable*) transitions that can (cannot) be prevented from firing by a *supervisory policy*. The supervisory policy decides which of the controllable transitions are permitted to fire at a marking reachable under supervision. The archetypal problem involves investigations

[1]Industrial & Enterprise Systems Engineering, and Coordinated Science Laboratory, University of Illinois at Urbana-Champaign {salimi2,somnath1,rsree}@illinois.edu.

into *liveness enforcing supervisory policies* (LESPs) that enforce liveness subsets of transitions in families of PNs that are not live (cf. [3] for a review).

The notion of *simulation* was introduced in the PN literature to formalize the concept of "similarity" among two PNs [4]. If a PN can simulate another PN, one could say that they are similar in some sense. This concept is generalized to include controlled PNs that are under the influence of a supervisory policy. We derive a necessary and sufficient condition for the existence of LESPs in PNs in a simulation relationship. The LESP for the simulated PN, along with the results on this paper and additional observations can oftentimes provide the LESP for the other PN. This is illustrated by an example, and we suggest further explorations into this method as a future research direction.

The rest of the paper is organized as follows. Section II introduces the relevant notations and reviews the results that are relevant to this paper. The main result is presented in section III, along with an example that demonstrates its utility in the synthesis of LESPs. The conclusions and suggested future research topics are presented in section IV.

II. Notations and Definitions and Some Preliminary Observations

We use $\mathcal{N}$ ($\mathcal{N}^+$) to denote the set of non-negative (positive) integers. The term $card(\bullet)$ denotes the cardinality of the set argument. A *Petri net structure* $N = (\Pi, T, \Phi, \Gamma)$ is an ordered 4-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, \ldots, t_m\}$ is a collection of m *transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, and $\Gamma : \Phi \to \mathcal{N}^+$ is the *weight* associated with each arc. The weight of an arc is represented by an integer that is placed along side the arc. For brevity, we refrain from denoting the weight of those arcs $\phi \in \Phi$ where $\Gamma(\phi) = 1$. A PN structure is said to be *ordinary* (*general*) if the weight associated with an arc is (not necessarily) unitary.

The *initial marking* of a PN structure N is a function $\mathbf{m}^0 : \Pi \to \mathcal{N}$, which identifies the number of *tokens* in each place. We will use the term *Petri net* (PN) and the symbol $N(\mathbf{m}^0)$ to denote a PN structure N along with its initial marking $\mathbf{m}^0$.

We define the sets $^\bullet x := \{y \mid (y, x) \in \Phi\}$ and $x^\bullet := \{y \mid (x, y) \in \Phi\}$. A transition $t \in T$ is said to be *enabled* at a marking $\mathbf{m}^i$ if $\forall p \in {^\bullet t}, \mathbf{m}^i(p) \geq \Gamma((p, t))$. The set of enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$. An enabled transition $t \in T_e(N, \mathbf{m}^i)$ can *fire*, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$ according to $\mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \Gamma(p, t) + \Gamma(t, p)$.

A. Supervisory Control of PNs

The paradigm of supervisory control of PNs assumes a subset of *controllable transitions*, denoted by $T_c \subseteq T$, which can be prevented from firing by an external agent called the *supervisor*. The set of *uncontrollable transitions*, denoted by $T_u \subseteq T$, is given by $T_u = T - T_c$. The controllable (uncontrollable) transitions are represented as filled (unfilled) boxes in graphical representation of PNs.

A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$, is a function that returns a 0 or 1 for each transition and each reachable marking. The supervisory policy $\mathcal{P}$ permits the firing of transition t_j at marking $\mathbf{m}^i$, only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $t_j \in T_e(N, \mathbf{m}^i)$ for some marking $\mathbf{m}^i$, we say the transition t_j is *state-enabled* at $\mathbf{m}^i$. If $\mathcal{P}(\mathbf{m}^i, t_j) = 1$, we say the transition t_j is *control-enabled* at $\mathbf{m}^i$. A transition has to be state- and control-enabled before it can fire. The fact that uncontrollable transitions cannot be prevented from firing by the supervisory policy is captured by the requirement that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$. This is implicitly assumed of any supervisory policy in this paper.

A string of transitions $\sigma = t_1 \cdots t_k$, where $t_j \in T (j \in \{1, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$, if, (1) $t_1 \in T_e(N, \mathbf{m}^i), \mathcal{P}(\mathbf{m}^i, t_1) = 1$, and (2) for $j \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{j+1}) = 1$.

The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$. If $\mathbf{m}^{i+k}$ results from the firing of $\sigma \in T^*$ starting from the initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^{i+k}$.

A transition t_k is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$.

Let $\mathcal{P}_0$ be the (trivial) supervisory policy that permits the firing of all controllable transitions at all markings. That is, $\forall t_c \in T_c, \forall \mathbf{m} \in \mathcal{N}^n, \mathcal{P}_0(\mathbf{m}, t_c) = 1$. In traditional PN theory concepts like valid firing strings and reachable marking are defined without reference to a supervisory policy. These concepts correspond to their counterparts introduced above under the policy $\mathcal{P}_0$. We will drop the reference to the supervisory policy when $\mathcal{P}_0$ is involved. For instance, if the statement "$\mathbf{m}^1 \xrightarrow{\sigma} \mathbf{m}^2$ in N" without explicit reference to a supervisory policy, we are implying that $\mathbf{m}^1 \xrightarrow{\sigma} \mathbf{m}^2$ in N under $\mathcal{P}_0$. Similarly, when we use the symbol $\Re(N, \mathbf{m}^0)$ in subsequent text, we are referring to $\Re(N, \mathbf{m}^0, \mathcal{P}_0)$.

If every transition in a subset $\widetilde{T} \subseteq T$ is live under the supervision of a policy $\mathcal{P}$ in $N(\mathbf{m}^0)$, we say the policy is a *$\widetilde{T}$-liveness enforcing supervisory policy* ($\widetilde{T}$-LESP). We drop the reference to the set $\widetilde{T}$ when $\widetilde{T} = T$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if for every $\widehat{T}$-LESP $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \to \{0, 1\}$ for $N(\mathbf{m}^0)$, the following condition holds $\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t)$. Continuing with the observations made in the previous paragraph, a PN $N(\mathbf{m}^0)$ is live in the traditional sense (i.e. in the absence of supervision) if $\mathcal{P}_0$ is an LESP for $N(\mathbf{m}^0)$.

For an arbitrary PN structure $N = (\Pi, T, \Phi, \Gamma)$, the set $\Delta(N) = \{\mathbf{m}^0 \in \mathcal{N}^{card(\Pi)} \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\}$ denotes the set of initial markings $\mathbf{m}^0$ for which there is a LESP for $N(\mathbf{m}^0)$. It follows that $\Delta(N)$ is *control invariant* with respect to N; that is, if $\mathbf{m}^1 \in \Delta(N), t_u \in T_e(N, \mathbf{m}^1) \cap T_u$ and $\mathbf{m}^1 \xrightarrow{t_u} \mathbf{m}^2$ in N, then $\mathbf{m}^2 \in \Delta(N)$. Equivalently, only the firing of a controllable transition at any marking in $\Delta(N)$ can result in a new marking that is not in $\Delta(N)$.

There is an LESP for $N(\mathbf{m}^0)$ if and only if $\mathbf{m}^0 \in \Delta(N)$.

If $\mathbf{m}^0 \in \Delta(N)$, the LESP that prevents the firing of a controllable transition at any marking when its firing would result in a new marking that is not in $\Delta(N)$, is the minimally restrictive LESP for $N(\mathbf{m}^0)$. The existence of an LESP for an arbitrary PN is undecidable in general, and is decidable if the PN structure N belongs to specific classes of PNs (cf. references [5], [6], [7], [8]).

We present a brief review of results that are pertinent to synthesis of LESPs for different classes of PNs in the literature. Giua [9] introduced *monitors* into supervisory control of PNs. Moody and Antsaklis [10] used monitors to enforce liveness in certain classes of PNs, this work was extended by Iordache and Antsaklis [11] to include a sufficient condition for the existence of policies that enforce liveness in a class of PNs called *Asymmetric Choice Petri nets*. Reveliotis et al. used the *theory of regions* to identify policies that enforce liveness in *Resource Allocation Systems* [12]. Ghaffari, Rezg and Xie [13] also used the theory of regions to obtain a *minimally restrictive* supervisory policy that enforces liveness for a class of PNs. Liu et al. [14] characterized the set of live initial markings of a class of general PN structures known as WS^3PR, which was used to construct monitors that enforce liveness in a class of WS^3PR. Marchetti and Munier-Kordon [15] presented a sufficient condition for liveness, that can be tested in polynomial time, for a class of general PNs known as *Unitary Weighted Event Graphs*. Basile et al. [16] presented sufficient conditions for minimally-restrictive, closed-loop liveness of a class of *Marked Graph* PNs supervised by monitors that enforce *Generalized Mutual Exclusion Constraints* (GMECs). Reference [17] presents a a necessary and sufficient condition for the existence of GMECs that enforces, among other things, liveness, in a bounded PN. Luo et al. [18] consider various properties of supervisory policies that ensure the marking of a PN stays within an appropriately defined polyhedron. Specifically, they consider the problem of eliminating redundant constraints in the polyhedron, which can be used to reduce the number of monitors in an invariant-based supervisor. Dideban et al. [19] use the concept of an *over-state* to reduce the number of monitors in an invariant-based controller for *safe* PNs. Chen et al. [20] use an Integer Linear Programming formulation along with Binary Decision Diagrams to obtain the GMEC parameters for deadlock avoidance using invariant-based monitors. Hu et al. [21] use *siphons* to characterize liveness in PN models of resource allocation systems, that are subsequently converted into a sequence of Mixed Integer Linear Programming instances.

The process of deciding the existence of an LESP in an arbitrary instance from these classes is NP-hard. In the following section, we extend the concept of *simulation* [4], originally introduced to capture the notion of "similarity" among traditional PNs, to PNs under supervision. This extended notion of "similarity under supervision" is used to derive a necessary and sufficient condition for the existence of LESPs (or, partial-LESPs) among PNs.

III. Main Results

We first extend Best's definition of *simulation* [4] to controlled PNs. Let $N = (\Pi, T, \Phi, \Gamma)$ and $\widehat{N} = (\widehat{\Pi}, \widehat{T}, \widehat{\Phi}, \widehat{\Gamma})$ be two PN structures. Suppose $\alpha : T \to \widehat{T}$ is an injection (i.e. every member of T has an image in $\widehat{T}$; but, the converse is not necessarily true). This injective function can be extended to include sets of transitions $T_1 \subseteq T$ as

$$\alpha(T_1) = \bigcup_{t \in T_1} \alpha(t).$$

With a slight abuse of notation, we extend the above function to strings of transitions $\alpha : T^* \to \widehat{T}^*$ for $\sigma \in T^*$ and $t \in T$, as

$$\alpha(\sigma t) \quad = \quad \alpha(\sigma)\alpha(t)$$

where $\alpha(\epsilon) = \epsilon$, and ϵ is the empty-string. The inverse-function $\alpha^{-1} : \widehat{T}^* \to T^*$ is defined for $\widehat{\sigma} \in \widehat{T}^*, \widehat{t} \in \widehat{T}$, as

$$\alpha^{-1}(\widehat{\sigma}\widehat{t}) = \alpha^{-1}(\widehat{\sigma}), \qquad \text{when } \widehat{t} \notin \alpha(T),$$
$$\alpha^{-1}(\widehat{\sigma}\widehat{t}) = \alpha^{-1}(\widehat{\sigma})\alpha^{-1}(\widehat{t}), \qquad \text{when } \widehat{t} \in \alpha(T),$$

and $\alpha^{-1}(\epsilon) = \epsilon$.

Definition 1: Let $N = (\Pi, T, \Phi, \Gamma)$ and $\widehat{N} = (\widehat{\Pi}, \widehat{T}, \widehat{\Phi}, \widehat{\Gamma})$ be two PN structures, and $\alpha : T \to \widehat{T}$ be an injective function. Suppose $N(\mathbf{m}^0)$ (resp. $\widehat{N}(\widehat{\mathbf{m}}^0)$) is supervised by policy $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \to \{0, 1\}$ (resp. $\widehat{\mathcal{P}} : \mathcal{N}^{card(\widehat{\Pi})} \times \widehat{T} \to \{0, 1\}$).
We say that $\widehat{\mathbf{m}}^0$ under the supervision of $\widehat{\mathcal{P}}$ *simulates* $N(\mathbf{m}^0)$ under the supervision of $\mathcal{P}$ if and only if there is a surjection $\beta : \mathcal{N}^{card(\widehat{\Pi})} \to \mathcal{N}^{card(\Pi)}$ such that:

1) $\mathbf{m}^0 = \beta(\widehat{\mathbf{m}}^0)$,
2) Suppose $\mathbf{m}^1 = \beta(\widehat{\mathbf{m}}^1), \widehat{\mathbf{m}}^1 \in \Re(\widehat{N}, \widehat{\mathbf{m}}^0, \widehat{\mathcal{P}})$ and $\mathbf{m}^1 \in \Re(N, \mathbf{m}^0, \mathcal{P})$, then
   a) whenever $\mathbf{m}^1 \xrightarrow{t} \mathbf{m}^2$ in N under $\mathcal{P}$, then $\exists \widehat{\mathbf{m}}^2 \in \beta^{-1}(\mathbf{m}^2), \exists \widehat{\sigma} \in \widehat{T}^*$ such that $\widehat{\mathbf{m}}^1 \xrightarrow{\widehat{\sigma}} \widehat{\mathbf{m}}^2$ under $\widehat{\mathcal{P}}$ in $\widehat{N}$, and $\alpha^{-1}(\widehat{\sigma}) = t$.
   b) whenever $\widehat{\mathbf{m}}^1 \xrightarrow{\widehat{\sigma}} \widehat{\mathbf{m}}^2$ under $\widehat{\mathcal{P}}$ in $\widehat{N}$ for some $\widehat{\sigma} \in \widehat{T}^*$, then $\mathbf{m}^1 \xrightarrow{\alpha^{-1}(\widehat{\sigma})} \beta(\widehat{\mathbf{m}}^2)$ under $\mathcal{P}$ in N.

As noted in reference [4], the fact that $\alpha : T \to \widehat{T}$ is an injective function would mean that $card(\widehat{T}) \geq card(T)$. The transitions in $\alpha(T)$ simulate the transitions in T, while the remaining transitions (i.e. the set $\widehat{T} - \alpha(T)$) are to be viewed as an internal to $\widehat{N}$.

A marking $\widehat{\mathbf{m}} \in \mathcal{N}^{card(\widehat{\Pi})}$ of $\widehat{N}$ represents the marking $\beta(\widehat{\mathbf{m}})$ in N. There could be many markings of $\widehat{N}$ can represent the a single marking of N. But the surjective nature of $\beta(\bullet)$ guarantees that every marking of N is represented by some marking of $\widehat{N}$.

Item 1 of definition 1 requires that the initial marking of $\widehat{N}$ must represent the initial marking of N. Item 2a requires that the firing of any transition t in N under the supervision of $\mathcal{P}$ must be simulated by the firing of a string of transitions in $\widehat{N}$ under the supervision of $\widehat{\mathcal{P}}$, while item 2b requires that every firing string in $\widehat{N}$ under the supervision of $\widehat{\mathcal{P}}$ has a corresponding firing string under the mapping $\alpha^{-1}(\bullet)$ that is permitted under $\mathcal{P}$ in N.

We now state and prove the main result of this paper.

Theorem 2: Suppose $N = (\Pi, T, \Phi, \Gamma)$ and $\widehat{N} = (\widehat{\Pi}, \widehat{T}, \widehat{\Phi}, \widehat{\Gamma})$ are two PN structures, and $\alpha : T \rightarrow \widehat{T}$ is an injective function. Let $N(\mathbf{m}^0)$ (resp. $\widehat{N}(\widehat{\mathbf{m}}^0)$) be supervised by policy $\mathcal{P} : \mathcal{N}^{card(\Pi)} \times T \rightarrow \{0, 1\}$ (resp. $\widehat{\mathcal{P}} : \mathcal{N}^{card(\widehat{\Pi})} \times \widehat{T} \rightarrow \{0, 1\}$), and $\widehat{N}(\widehat{\mathbf{m}}^0)$ under the supervision of $\widehat{\mathcal{P}}$ simulates $N(\mathbf{m}^0)$ under $\mathcal{P}$ with respect to $\alpha(\bullet)$, then $\mathcal{P}$ is an LESP for $N(\mathbf{m}^0)$ if and only if $\forall \widehat{t} \in \alpha(T)$, $\widehat{t}$ is live under $\widehat{\mathcal{P}}$.

Proof: (*Only If*) Let $\mathcal{P}$ be an LESP for $N(\mathbf{m}^0)$, and $\widehat{\mathbf{m}}^0 \xrightarrow{\widehat{\sigma}_1} \widehat{\mathbf{m}}^1$ under $\widehat{\mathcal{P}}$ in $\widehat{N}(\widehat{\mathbf{m}}^0)$. From item 1 of definition 1 we have $\mathbf{m}^0 = \beta(\widehat{\mathbf{m}}^0)$. From item 2b, $\mathbf{m}^0 \xrightarrow{\alpha^{-1}(\widehat{\sigma}_1)} \beta(\widehat{\mathbf{m}}^1)$ under $\mathcal{P}$ in N.

Since $\mathcal{P}$ is an LESP for N, $\forall t \in T, \exists \sigma_2 \in T^*, \exists \mathbf{m}^2 \in \mathcal{N}^{card(\Pi)}$ such that $\beta(\widehat{\mathbf{m}}^1) \xrightarrow{\sigma_2} \mathbf{m}^2$ in N under $\mathcal{P}$, where t occurs once in σ_2. From item 2a of definition 1, $\exists \widehat{\sigma}_2 \in \widehat{T}^*, \exists \widehat{\mathbf{m}}^2 \in \Re(\widehat{N}, \widehat{\mathbf{m}}^0, \widehat{\mathcal{P}})$ such that $\widehat{\mathbf{m}}^1 \xrightarrow{\widehat{\sigma}_2} \widehat{\mathbf{m}}^2$ and $\alpha(t)$ occurs in $\widehat{\sigma}_2$. Therefore, all $\widehat{t} \in \alpha(T)$ is live under $\widehat{\mathcal{P}}$ in $\widehat{N}(\widehat{\mathbf{m}}^0)$.

(*If*) Suppose, $\forall \widehat{t} \in \alpha(T), \widehat{t}$ is live in $\widehat{N}$ under $\widehat{\mathcal{P}}$. Let $\mathbf{m}^0 \xrightarrow{\sigma_1} \mathbf{m}^1$ under $\mathcal{P}$ in N, where $\sigma_1 = t_1 \cdots t_m$. From items 1 and 2a of definition 1, $\exists \widehat{\sigma}_1, \ldots, \widehat{\sigma}_m \in \widehat{T}^*$ such that $\widehat{\mathbf{m}}^0 \xrightarrow{\widehat{\sigma}_1 \cdots \widehat{\sigma}_m} \widehat{\mathbf{m}}^1$ in $\widehat{N}$ under the supervision of $\widehat{\mathcal{P}}$, and $\mathbf{m}^1 = \beta(\widehat{\mathbf{m}}^1)$.

Since all transitions in $\alpha(T)$ are live under $\widehat{\mathcal{P}}$ in $\widehat{N}$, $\exists \widehat{\sigma}_2 \in \widehat{T}^*, \exists \widehat{\mathbf{m}}^2 \in \mathcal{N}^{card(\widehat{\Pi})}$, such that $\widehat{\mathbf{m}}^1 \xrightarrow{\widehat{\sigma}_2} \widehat{\mathbf{m}}^2$ in $\widehat{N}$ under $\widehat{\mathcal{P}}$, and $\alpha(t)$ occurs in $\widehat{\sigma}_2$, for any $t \in T$. By item 2b of definition 1, $\mathbf{m}^1 \xrightarrow{\alpha^{-1}(\widehat{\sigma}_2)} \mathbf{m}^2$ under $\mathcal{P}$ in N, where $\mathbf{m}^2 = \beta(\widehat{\mathbf{m}}^2)$ and $t \in T$ occurs in $\alpha^{-1}(\widehat{\sigma}_2)$. Therefore, every transition in T is live under the supervision of $\mathcal{P}$ in $N(\mathbf{m}^0)$. ■

The above result notes that the liveness of the transitions in the set T of the PN N under $\mathcal{P}$ is equivalent to the liveness of the transitions $\alpha(T)$ in the PN $\widehat{N}$ under $\widehat{\mathcal{P}}$. It is possible that some of the transitions in the set $\widehat{T} - \alpha(T)$ are not live under $\widehat{\mathcal{P}}$ in $\widehat{N}$. However, if $\widehat{N}$ has a structure that permits us to infer the liveness of the set $\widehat{T} - \alpha(T)$ from the liveness of $\alpha(T) \subseteq \widehat{T}$, then theorem 2 can be enhanced to a result that notes $\mathcal{P}$ is an LESP for $N(\mathbf{m}^0)$ if and only if $\widehat{\mathcal{P}}$ is an LESP for $\widehat{N}$. We illustrate this concept using an example, and suggest further exploration of this theme as a future research topic.

Consider the PN structures $N = (\Pi, T, \Phi, \Gamma)$ and $\widehat{N} = (\widehat{\Pi}, \widehat{T}, \widehat{\Phi}, \widehat{\Gamma})$ shown in figure 1(a) and (b) respectively. The injective function $\alpha : T \rightarrow \widehat{T}$ is defined in table I.

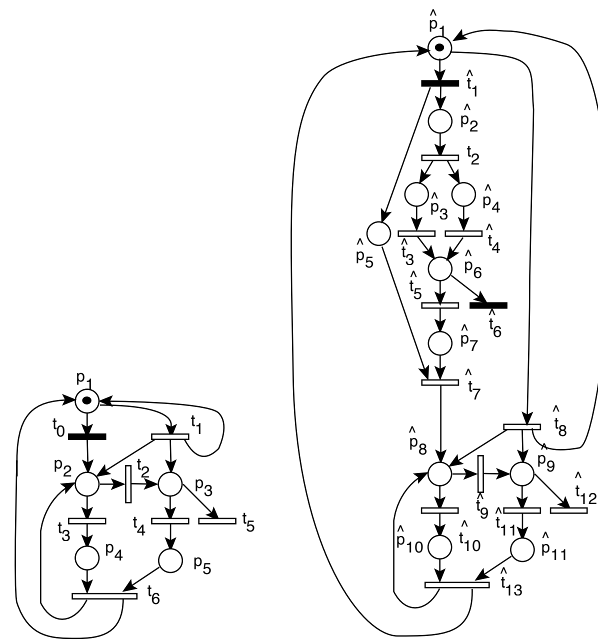
The surjective function $\beta : \mathcal{N}^{11} \rightarrow \mathcal{N}^5$ for N and $\widehat{N}$ of figure 1(a) and (b) is given by the function $\beta(\mathbf{m}) = (\mathbf{m}_1 \;\; \mathbf{m}_8 \;\; \mathbf{m}_9 \;\; \mathbf{m}_{10} \;\; \mathbf{m}_{11})^T$. The token load in places $\widehat{p}_1, \widehat{p}_8, \widehat{p}_9, \widehat{p}_{10}$ and $\widehat{p}_{11}$ correspond to the token load in places p_1, p_2, p_3, p_4 and p_5 respectively.

Let $\widehat{\mathcal{P}}$ be any policy that permits the firing of $\widehat{t}_6$ if and only if $((\widehat{\mathbf{m}}_1 + \widehat{\mathbf{m}}_2 + \widehat{\mathbf{m}}_3 + \widehat{\mathbf{m}}_4 + \widehat{\mathbf{m}}_6 + \widehat{\mathbf{m}}_7) \geq 2)$. Let us also suppose the $\mathcal{P}$ permits the firing of t_0 at marking

$t \in T$	$\alpha(t) \in \widehat{T}$
t_0	$\widehat{t}_1$
t_1	$\widehat{t}_8$
t_2	$\widehat{t}_9$
t_3	$\widehat{t}_{10}$
t_4	$\widehat{t}_{11}$
t_5	$\widehat{t}_{12}$
t_6	$\widehat{t}_{13}$

TABLE I

THE INJECTIVE FUNCTION $\alpha : T \rightarrow \widehat{T}$ FOR THE PNS SHOWN IN FIGURE 1(A) AND 1(B).



(a) $N(\mathbf{m}^0)$ (b) $\widehat{N}(\widehat{\mathbf{m}}^0)$

Fig. 1. If $\widehat{\mathcal{P}}$ is a policy that permits the firing of $\widehat{t}_6$ if and only if $((\widehat{\mathbf{m}}_1 + \widehat{\mathbf{m}}_2 + \widehat{\mathbf{m}}_3 + \widehat{\mathbf{m}}_4 + \widehat{\mathbf{m}}_6 + \widehat{\mathbf{m}}_7) \geq 2)$; and, permits the firing of $\widehat{t}_1 (= \alpha(t_0))$ at every marking in $\beta^{-1}(\mathbf{m})$ if and only if $\mathcal{P}$ permits the firing of t_0 at marking $\mathbf{m}$, then $\widehat{N}(\widehat{\mathbf{m}}^0)$ under $\widehat{\mathcal{P}}$ *simulates* $N(\mathbf{m}^0)$ under $\mathcal{P}$. From Theorem 2 we infer that the policy $\mathcal{P}$ is an LESP for $N(\mathbf{m}^0)$ if and only if every transition in $\alpha(T) = \{\widehat{t}_1, \widehat{t}_8, \widehat{t}_9, \widehat{t}_{10}, \widehat{t}_{11}, \widehat{t}_{12}, \widehat{t}_{13}\}$ is live under $\widehat{\mathcal{P}}$ in $\widehat{N}$.

$\mathbf{m} \in \mathcal{N}^5$ if and only if $\widehat{\mathcal{P}}$ permits the firing of $\widehat{t}_1 (= \alpha(t_0))$ at all markings in $\beta^{-1}(\mathbf{m})$. Then, $\widehat{N}(\widehat{\mathbf{m}}^0)$ under $\widehat{\mathcal{P}}$ simulates $N(\mathbf{m}^0)$ under $\mathcal{P}$. Consequently, by Theorem 2, $\mathcal{P}$ is an LESP for $N(\mathbf{m}^0)$ if and only if every transition in $\alpha(T) = \{\widehat{t}_1, \widehat{t}_8, \widehat{t}_9, \widehat{t}_{10}, \widehat{t}_{11}, \widehat{t}_{12}, \widehat{t}_{13}\}$ is live under $\widehat{\mathcal{P}}$ in $\widehat{N}$.

From the structure of $\widehat{N}$ we can infer that the liveness of $\widehat{t}_1$ implies the liveness of the transitions in the set $\widehat{T} - \alpha(T) = \{\widehat{t}_2, \widehat{t}_3, \widehat{t}_4, \widehat{t}_5, \widehat{t}_6, \widehat{t}_7\}$. Consequently, $\mathcal{P}$ is an LESP for $N(\mathbf{m}^0)$ if and only if $\widehat{\mathcal{P}}$ is an LESP for $\widehat{N}(\widehat{\mathbf{m}}^0)$.

From method of reference [22], we know that the supervisory policy $\mathcal{P}$ that permits the firing of t_0 if and only if $((\mathbf{m}_1 \geq 2) \vee (\mathbf{m}_4 \geq 1) \wedge (\mathbf{m}_5 \geq 2))$ is an LESP. As a consequence of the above observation, we can conclude that the supervisory policy $\widehat{\mathcal{P}}$ that permits the firing of $\widehat{t}_1$ if and only if $((\widehat{\mathbf{m}}_1 \geq 2) \vee (\widehat{\mathbf{m}}_{10} \geq 1) \wedge (\widehat{\mathbf{m}}_{11} \geq 2))$; and permits the firing of $\widehat{t_6}$ if and only if $((\widehat{\mathbf{m}}_1 + \widehat{\mathbf{m}}_2 + \widehat{\mathbf{m}}_3 + \widehat{\mathbf{m}}_4 + \widehat{\mathbf{m}}_6 + \widehat{\mathbf{m}}_7) \geq 2)$ is an LESP for $\widehat{N}(\mathbf{m}^0)$. That is, the LESP for the larger PN $\widehat{N}(\mathbf{m}^0)$ was synthesized from the LESP for the smaller PN $N(\mathbf{m}^0)$ with the help of the results in this paper.

The above observation used the structure of $\widehat{N}$ to conclude that the liveness of the set of transitions $\alpha(T)$ under a supervisory policy implies the liveness of the transitions in $\widehat{T} - \alpha(T)$ as well. We suggest the identification of general conditions that are sufficient to make this inference on a wider class of PNs as a future research topic. This is hoped that this will mitigate the computational effort that is required in the synthesis of LESPs for large PN models.

IV. Conclusions

In this paper we extended the notion of *simulation*, introduced to capture "similarity" among different *Petri net* (PN) models, to *controlled* PNs [4]. The set of transitions in a controlled PN are partitioned into *controllable*- and *uncontrollable* subsets. The firing of a controllable transition requires (1) the presence of sufficient tokens in its input places, and (2) the permission of an external supervisory policy. The supervisor policy is usually modeled as a static-map that returns a 0 or a 1 for each marking and each controllable transition. An uncontrollable transition, which represents an activity that is external to the system, cannot be prevented from firing by a supervisory policy – it can potentially fire when there are sufficient tokens in its input places.

A transition in a PN model is *live* under supervision if it can be fired, not necessarily immediately, from every marking that is reachable under supervision. It is of interest to synthesize supervisory policies that enforce liveness in a subset of transitions in a controlled PN.

We formally defined the conditions under which a PN $\widehat{N}(\widehat{\mathbf{m}}^0)$ under a supervisory policy $\widehat{\mathcal{P}}$ simulates a PN $N(\mathbf{m}^0)$ under a policy $\mathcal{P}$. Under these conditions, we showed that all transitions in the PN $N(\mathbf{m}^0)$ are live under the policy $\mathcal{P}$ if and only if an appropriate subset of transitions of $\widehat{N}(\widehat{\mathbf{m}}^0)$ are live under the policy $\widehat{\mathcal{P}}$. Using an illustrative example, we showed how this result, along with some problem specific observations, could be used to synthesize LESPs in larger PNs.

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