# On a Sufficient Information Structure for Supervisory Policies that Enforce Liveness in a Class of General Petri Nets

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***Abstract*—A *Petri net* (PN) is said to be *live* if it is possible to fire any transition, although not immediately, from every reachable marking. A *liveness enforcing supervisory policy* (LESP) determines which controllable transition is to be prevented from firing at a marking, to ensure the supervised *Petri net* (PN) is live.**

**In this paper, we restrict our attention to a class of general *Petri nets* (PN) structures, where the existence of an LESP for an instance initialized at a marking, implies the existence of an LESP when the same instance is initialized with a termwise-larger initial marking. We show that the minimally restrictive LESP for an instance $N$ from this class is characterized by a collection of boolean formulae $\{\Theta_{t_c}(N)\}_{t_c \in T_c}$, where $T_c$ is the set of controllable transitions in the PN. The literals in $\Theta_{t_c}(N)$ are true if and only if the token-load of specific places meet a threshold. Consequently, appropriately placed *threshold-sensors*, which detect if the token-load of a place is greater than or equal to a predetermined threshold, provide sufficient information to implement the minimally restrictive LESP. The paper concludes with some directions for future research.**

***Index Terms*—Petri Nets, Supervisory Control.**

## I. Introduction

A *Petri net* (PN) is *live* if, irrespective of the past transition-firings, all transitions can be fired subsequently. A live PN does not experience deadlocks, but a deadlock-free PN is not necessarily live. A PN that is not live can be made live by a *liveness enforcing supervisory policy* (LESP) that determines which controllable transition is to be disabled from firing at a given marking, such that the supervised PN is live. This paradigm assumes that only a subset of the set of transitions in a PN can be prevented from firing by the LESP.

An LESP $\mathcal{P}_N^*$ for a PN $N(\mathbf{m}^0)$ is said to be *minimally restrictive* if the fact that a controllable transition is prevented from firing at a marking by $\mathcal{P}_N^*$ implies that all other LESPs for $N(\mathbf{m}^0)$, irrespective of the implementation paradigm, will prevent its firing at the same marking. The existence of an LESP for a PN $N(\mathbf{m}^0)$ implies the existence of a unique minimally restrictive LESP for $N(\mathbf{m}^0)$, which is defined in terms of the set of markings

$$\Delta(N) = \{\mathbf{m}^0 \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\}.$$

There is an LESP for $N(\mathbf{m}^0)$ if and only if $\mathbf{m}^0 \in \Delta(N)$. The minimally restrictive LESP $\mathcal{P}_N^*$ prevents the firing of a controllable transition at a marking in $\Delta(N)$ if and only if its firing would result in a new marking that is not in $\Delta(N)$.

In this paper, we restrict our attention to PNs for which $\Delta(N)$ is *right-closed*. That is, the existence of a marking in $\Delta(N)$ implies the existence of all termwise-larger markings in $\Delta(N)$. This class of PNs includes the class of Ordinary Free-Choice PNs, which has found extensive use in modeling product-flow in manufacturing systems (cf. chapter 2, [1]), and control-flow in processor networks (cf. section 1.2, [2]). For this class of PNs, the set $\Delta(N)$ is identified by a finite set of minimal elements $\min(\Delta(N))$. In a naive implementation, the minimally restrictive LESP $\mathcal{P}_N^*$ would involve testing if the new marking that will result from the firing of a controllable transition is greater than or equal to one of the members of $\min(\Delta(N))$. We show that $\mathcal{P}_N^*$ can be equivalently represented by a collection of *Disjunctive Normal Form* (DNF) expressions $\{\Theta_{t_c}(N)\}_{t_c \in T_c}$, where $T_c$ denotes the set of controllable transitions. Furthermore, the policy of permitting the controllable transition $t_c \in T_c$ at a marking if and only if the DNF-expression $\Theta_{t_c}(N)$ is satisfied by the marking, is equivalent to the minimally restrictive LESP $\mathcal{P}_N^*$. We note that the literals in the DNF-expressions are of the form "$\mathbf{m}(p) \geq \beta$," which in turn implies that a *threshold-sensor* at appropriate places provides sufficient information for enforcing the minimally restrictive LESP $\mathcal{P}_N^*$. A $\beta$-threshold sensor at a place senses if the token-load of the place is greater than or equal to $\beta$. That is, the minimally restrictive LESP can be implemented as a static-map that requires *just* threshold sensors, located at select places. It is not necessary to sense the firing of transitions, nor is it necessary to measure the exact number of tokens in select places, as would perhaps be the case with other constructions that enforce liveness.

We suggest explorations into using fault-tolerant implementations of $\mathcal{P}_N^*$ as a future research topic. We present a brief review of relevant prior work in the next subsection.

### A. Review of Relevant Prior Work

*Monitors* are places added to an existing PN structure, whose token-load at any instant indicates the amount of a particular resource that is available for consumption. The input and output arcs to this place appropriately capture the consumption and production of resources in the original PN. These were originally introduced into supervisory control of PNs by Giua [3] to handle mutual exclusion constraints. Moody and Antsaklis represent liveness constraints in specific PNs as linear inequalities, which are then implemented using monitor places. This work was extended by Iordache and Antsaklis to include a sufficient condition for the existence of policies that enforce liveness in a class of PNs called *Asymmetric Choice Petri nets*[1] [5].

Structural features of a PN, known as *siphons*, characterize the liveness of some classes of PNs. Several authors have used monitor place constructions that prevent siphons from being undermarked (cf. [6], [7], for example). References [8], [9] use a set of inequalities to characterize insufficiently marked siphons that is subsequently used to develop an algebraic LESP-synthesis procedure. Li et al [10] develop an iterative siphon-based control scheme for preventing deadlocks in PN models of manufacturing systems using a mixed integer programming approach involving what are known as *necessary siphons*.

The attractive feature of monitor-based supervision has been that the control effected by the supervisory policy uses the PN firing-mechanism. Unfortunately, if and when they exist, monitor-based LESPs are not guaranteed to be minimally restrictive in every instance (cf. reference [11] for details). In contrast, the work presented in this paper provides an LESP that *is* minimally restrictive, and uses an information structure that only involves boolean-expressions that use the outputs of threshold sensors in selected places.

Reveliotis [12] developed a class of policies for resource allocation systems that can be extended to the PN-framework using the *theory of regions*. Ghaffari, Rezg and Xie [13] also use the theory of regions to obtain a minimally restrictive supervisory policy that enforces liveness for a class of PNs. Marchetti and Munier-Kordon [14] presented a sufficient condition for liveness, that can be tested in polynomial time, for a class of general PNs known as *Unitary Weighted Event Graphs*. Basile et al. [15] presented sufficient conditions for minimally-restrictive, closed-loop liveness of a class of *Marked Graph* PNs supervised by monitors that enforce *Generalized Mutual Exclusion Constraints* (GMECs).

The rest of the paper is organized as follows. After introducing the notations and definitions, section II introduces the paradigm of supervisory control of PNs, and presents a brief review of relevant prior work. The main results of the paper are presented in section III. The conclusions and suggested future research directions are presented in section IV.

## II. Notations and Definitions and Some Preliminary Observations

We use $\mathcal{N}$ ($\mathcal{N}^+$) to denote the set of non-negative (positive) integers. The term *card*($\bullet$) denotes the cardinality of the set argument. A *Petri net structure* $N = (\Pi, T, \Phi, \Gamma)$ is an ordered 4-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of *n places*, $T = \{t_1, \ldots, t_m\}$ is a collection of *m transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, and $\Gamma : \Phi \to \mathcal{N}^+$ is the *weight* associated with each arc. The weight of an arc is represented by an integer that is placed alongside the arc. For brevity, we refrain from denoting the weight of those arcs $\phi \in \Phi$ where $\Gamma(\phi) = 1$. A PN structure is said to be *ordinary* (*general*) if the weight associated with an arc is (not necessarily) unitary.

The *initial marking function* (or the *initial marking*) of a PN structure $N$ is a function $\mathbf{m}^0 : \Pi \to \mathcal{N}$, which identifies the number of *tokens* in each place. We will use the term *Petri net* (PN) and the symbol $N(\mathbf{m}^0)$ to denote a PN structure N along with its initial marking $\mathbf{m}^0$.

A marking $\mathbf{m} : \Pi \to \mathcal{N}$ is sometimes represented by an integer-valued vector $\mathbf{m} \in \mathcal{N}^n$, where the $i$-th component $\mathbf{m}_i$ represents the token-load ($\mathbf{m}(p_i)$) of the $i$-th place.

We define the sets ${}^\bullet x := \{y \mid (y, x) \in \Phi\}$ and $x^\bullet := \{y \mid (x, y) \in \Phi\}$. A transition $t \in T$ is said to be *enabled* at a marking $\mathbf{m}^i$ if $\forall p \in {}^\bullet t, \mathbf{m}^i(p) \geq \Gamma(p, t)$. The set of enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$. An enabled transition $t \in T_e(N, \mathbf{m}^i)$ can *fire*, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$ according to $\mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \Gamma(p, t) + \Gamma(t, p)$.

When the marking is interpreted as a nonnegative integer-valued vector, it is useful to define the *input matrix* $\mathbf{IN}$ (*output matrix* $\mathbf{OUT}$) as an $n \times m$ matrix, where $\mathbf{IN}_{i,j}$ ($\mathbf{OUT}_{i,j}$) equals $\Gamma((p_i, t_j))$ ($\Gamma((p_i, t_j))$) if $p_i \in {}^\bullet t_j$, ($p_i \in t_j^\bullet$) and is zero-valued otherwise. The *incidence matrix* $\mathbf{C}$ of the PN $N$ is an $n \times m$ matrix, where $\mathbf{C} = \mathbf{OUT} - \mathbf{IN}$.

A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* [16] if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \geq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 \in \mathcal{M}))$. The set $\mathcal{M} \subseteq \mathcal{N}^n$ contains a finite set of minimal-elements, $min(\mathcal{M}) \subset \mathcal{M}$, and is uniquely determined by them.

### A. Supervisory Control of PNs

The paradigm of supervisory control of PNs assumes a subset of *controllable transitions*, denoted by $T_c \subseteq T$, which can be prevented from firing by an external agent called the *supervisor*. The set of *uncontrollable transitions*, denoted by $T_u \subseteq T$, is given by $T_u = T - T_c$. The controllable (uncontrollable) transitions are represented as filled (unfilled) boxes in graphical representation of PNs.

A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$, is a function that returns a 0 or 1 for each transition and each reachable marking. The supervisory policy $\mathcal{P}$ permits the firing of transition $t_j$ at marking $\mathbf{m}^i$, only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $t_j \in T_e(N, \mathbf{m}^i)$ for some marking $\mathbf{m}^i$, we say the transition $t_j$ is *state-enabled* at $\mathbf{m}^i$. If $\mathcal{P}(\mathbf{m}^i, t_j) = 1$, we say the transition $t_j$ is *control-enabled* at $\mathbf{m}^i$. A transition has to be state- and control-enabled before it can fire. The fact that uncontrollable transitions cannot be prevented from firing by the supervisory policy is captured by the requirement that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$. This is implicitly assumed of any supervisory policy in this paper.

A string of transitions $\sigma = t_{i_1} t_{i_2} \cdots t_{i_k}$, where $t_{i_j} \in T (j \in \{1, 2, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$, if, (1) $t_{i_1} \in T_e(N, \mathbf{m}^i), \mathcal{P}(\mathbf{m}^i, t_{i_1}) = 1$, and (2) for $j \in \{1, 2, \ldots, k-1\}$ the firing of the transition $t_{i_j}$ produces a marking $\mathbf{m}^{i+j}$ and $t_{i_{j+1}} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{i_{j+1}}) = 1$.

The set of reachable markings under the supervision of $\mathcal{P}$ in $N$ from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$. If $\mathbf{m}^{i+k}$ results from the firing of $\sigma \in T^*$ starting from the

[1]cf. page 554, [4] for a formal definition.

initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^{i+k}$. If $\mathbf{x}(\sigma)$ is an $m$-dimensional vector whose $i$-th component corresponds to the number of occurrences of $t_i$ in a valid string $\sigma \in T^*$, and if $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^{i+j}$, then $\mathbf{m}^{i+j} = \mathbf{m}^i + \mathbf{C}\mathbf{x}(\sigma)$.

A transition $t_k$ is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$.

A policy $\mathcal{P}$ is a *liveness enforcing supervisory policy* (LESP) for $N(\mathbf{m}^0)$ if all transitions in $N(\mathbf{m}^0)$ are live under $\mathcal{P}$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if for every LESP $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \rightarrow \{0, 1\}$ for $N(\mathbf{m}^0)$, the following condition holds $\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t)$.

For an arbitrary PN structure $N = (\Pi, T, \Phi, \Gamma)$, the set

$$\Delta(N) = \{\mathbf{m}^0 \in \mathcal{N}^{card(\Pi)} \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\}$$

denotes the set of initial markings $\mathbf{m}^0$ for which there is an LESP for $N(\mathbf{m}^0)$. $\Delta(N)$ is *control invariant* (cf. proposition 7.1, [17]) with respect to $N$; that is, if $\mathbf{m}^1 \in \Delta(N), t_u \in T_e(N, \mathbf{m}^1) \cap T_u$ and $\mathbf{m}^1 \xrightarrow{t_u} \mathbf{m}^2$ in $N$, then $\mathbf{m}^2 \in \Delta(N)$. Equivalently, only the firing of a controllable transition at any marking in $\Delta(N)$ can result in a new marking that is not in $\Delta(N)$.

There is an LESP for $N(\mathbf{m}^0)$ if and only if $\mathbf{m}^0 \in \Delta(N)$. If $\mathbf{m}^0 \in \Delta(N)$, the LESP that prevents the firing of a controllable transition at any marking when its firing would result in a new marking that is not in $\Delta(N)$, is the minimally restrictive LESP for $N(\mathbf{m}^0)$ (cf. Lemma 5.9, [18]). We use the symbol $\mathcal{P}^*_N$ to denote this minimaly restrictive LESP. The existence of an LESP for an arbitrary PN is undecidable (cf. corollary 5.2, [19]), and is decidable if all transitions in the PN are controllable, or if the PN structure $N$ belongs to the classes identified in the following references [18], [20], [21]. The process of deciding the existence of an LESP in an arbitrary instance from these classes is NP-hard.

We present the main results of this paper in the next section.

## III. Main Results

As noted earlier, for any PN $N(\mathbf{m}^0)$, where $\mathbf{m}^0 \in \Delta(N)$, the minimally restrictive LESP $\mathcal{P}^*_N$ disables a controllable transition $t_c \in T_c$ at a marking $\mathbf{m}^1 \in \Re(N, \mathbf{m}^0, \mathcal{P}^*_N)$ if and only if $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$ in $N(\mathbf{m}^1)$ and $\mathbf{m}^2 \notin \Delta(N)$.

For the remainder of this paper, we restrict attention to PN structures $N$, where $\Delta(N)$ is right-closed. The finite set of minimal elements of $\Delta(N)$, $\min(\Delta(N))$, can be computed for this class of PN structures (cf. [22], [23], [24]). The remainder of this paper is about characterizing the minimally restrictive LESP $\mathcal{P}^*_N$ using boolean expressions that involve inequalities on the token-loads of places, which in turn yields a sufficient sensing strategy for the implementation of the minimally restrictive LESP.

The following observation identifies the controllable transitions that are never control-disabled by the minimally restrictive LESP $\mathcal{P}^*_N$.

**Lemma III.1.** *Let $N = (\Pi, T, \Phi, \Gamma)$ be a PN structure where $\Delta(N)$ is right-closed. Suppose $\min(\Delta(N)) = \{\widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_2, \ldots, \widehat{\mathbf{m}}_k\}$, where each $\widehat{\mathbf{m}}_i \in \mathcal{N}^n$ and $card(\Pi) = n$. A controllable transition $t_c \in T_c$ is control-enabled by the minimally restrictive LESP $\mathcal{P}^*_N$ at any marking in $\Delta(N)$ if and only if $\forall \widehat{\mathbf{m}}_i \in \min(\Delta(N))$, $\exists \widehat{\mathbf{m}}_j \in \min(\Delta(N))$, such that*

$$\max\{\mathbf{IN}_{\bullet,c}, \widehat{\mathbf{m}}_i\} + \mathbf{C} \times \mathbf{1}_c \geq \widehat{\mathbf{m}}_j,$$

*where $\mathbf{IN}_{\bullet,c}$ denotes the c-th column of the input matrix $\mathbf{IN}$, $\mathbf{1}_c$ is the unit-vector where the c-th element is unity, and the* max*-operation is done element-wise.*

*Proof.* (*If*) Suppose for some $\widehat{\mathbf{m}}_i \in \min(\Delta(N))$, $\exists \widehat{\mathbf{m}}_j \in \min(\Delta(N))$, such that

$$\max\{\mathbf{IN}_{\bullet,c}, \widehat{\mathbf{m}}_i\} + \mathbf{C} \times \mathbf{1}_c \geq \widehat{\mathbf{m}}_j.$$

If $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$ and $\mathbf{m}^1 \geq \widehat{\mathbf{m}}_i (\Rightarrow \mathbf{m}^1 \in \Delta(N))$, then $\mathbf{m}^2 \in \Delta(N)$. Consequently, $\mathcal{P}^*_N$ will not control-disable $t_c$ at the marking $\mathbf{m}^1$. The result follows from the fact that that above claim holds $\forall \widehat{\mathbf{m}}_i \in \min(\Delta(N))$.

(*Only If*) Suppose $\exists \widehat{\mathbf{m}}_i \in \min(\Delta(N))$, such that $\forall \widehat{\mathbf{m}}_j \in \min(\Delta(N))$,

$$\max\{\mathbf{IN}(\bullet, c), \widehat{\mathbf{m}}_i\} + \mathbf{C} \times \mathbf{1}_c \ngeq \widehat{\mathbf{m}}_j.$$

Then, for marking $\mathbf{m}^1 = \max\{\mathbf{IN}(\bullet, c), \widehat{\mathbf{m}}_i\} (\Rightarrow \mathbf{m}^1 \in \Delta(N))$, the minimally restrictive LESP $\mathcal{P}^*_N$ will control disable transition $t_c \in T_c$ at marking $\mathbf{m}^1 \in \Delta(N)$. □

To illustrate lemma III.1, we consider the *Free Choice* PN (FCPN) (cf. section VI-A, [4] for a formal definition) $N_1$ structure shown in figure 1(a). We have $\widehat{\mathbf{m}}_1 = (1\ 0\ 0\ 0\ 0)^T$ and $\widehat{\mathbf{m}}_2 = (0\ 0\ 0\ 1\ 1)^T$. For transition $t_3 \in T_c$, we note that the conditions of lemma III.1 is satisfied as

$$\max\left\{\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\right\} + \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} \geq \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \text{and}$$

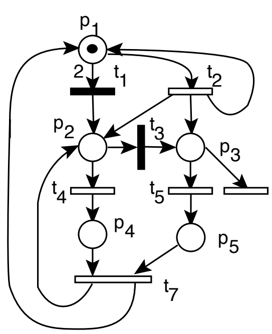
$$\max\left\{\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}\right\} + \begin{pmatrix}0\\-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix} \geq \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}.$$

Consequently, transition $t_3$ is never control-disabled by the minimally restrictive LESP for $N_1(\mathbf{m}^0_1)$.

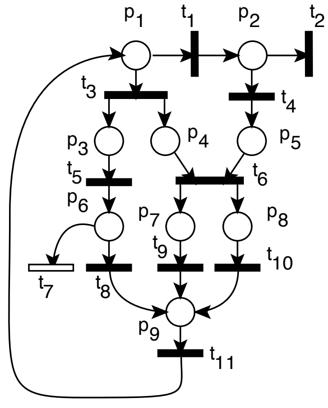
As an additional illustration, the controllable transitions $t_4, t_5, t_6, t_8, t_9, t_{10}$ and $t_{11}$ in the PN structure $N_2$ shown in figure 1(b), meet the requirement of lemma III.1 (cf. figure 1(c), which lists the members of $\min(\Delta(N_2))$. Consequently, these controllable transitions are never control-disabled by a minimally restrictive LESP for $N_2(\mathbf{m}^0_2)$.

On the flip-side, the controllable transition $t_1$ of $N_1$ does not meet the requirements of lemma III.1. It is control-disabled by the minimally restrictive LESP for $N_1(\mathbf{m}^0_1)$ for $\mathbf{m}^0_1 = (2\ 0\ 0\ 0\ 0)^T (\in \Delta(N_1))$. Likewise, the controllable transitions $t_1$ and $t_2$ will be control-disabled by the minimally restrictive LESP for $N_2(\mathbf{m}^0_2)$ for $\mathbf{m}^0_2 = (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^T (\in \Delta(N_2))$; $t_3$ is control-disabled by the minimally restrictive LESP for $N_2(\mathbf{m}^0_2)$ for $\mathbf{m}^0_2 = (1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0)^T (\in \Delta(N_2))$.

We turn our attention to those controllable transitions $t_c \in T_c$, that do not satisfy the requirement of lemma III.1, and




(a) General FCPN structure $N_1$




(b) FCPN structure $N_2$

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 1: ( 0 0 0 1 1 0 0 0 0 )
 2: ( 0 1 0 1 0 0 0 0 0 )
 3: ( 2 0 0 0 0 0 0 0 0 )
 4: ( 1 1 0 0 0 0 0 0 0 )
 5: ( 1 0 0 1 0 0 0 0 0 )
 6: ( 1 0 0 0 1 0 0 0 0 )
 7: ( 1 0 0 0 0 0 1 0 0 )
 8: ( 1 0 0 0 0 0 0 1 0 )
 9: ( 1 0 0 0 0 0 0 0 1 )
10: ( 0 1 0 0 0 0 1 0 0 )
11: ( 0 0 0 1 0 0 1 0 0 )
12: ( 0 0 0 0 1 0 1 0 0 )
13: ( 0 0 0 0 0 0 2 0 0 )
14: ( 0 0 0 0 0 0 1 1 0 )
15: ( 0 0 0 0 0 0 1 0 1 )
16: ( 0 1 0 0 0 0 0 1 0 )
17: ( 0 0 0 1 0 0 0 1 0 )
18: ( 0 0 0 0 1 0 0 1 0 )
19: ( 0 0 0 0 0 0 0 2 0 )
20: ( 0 0 0 0 0 0 0 1 1 )
21: ( 0 1 0 0 0 0 0 0 1 )
22: ( 0 0 0 1 0 0 0 0 1 )
23: ( 0 0 0 0 1 0 0 0 1 )
24: ( 0 0 0 0 0 0 0 0 2 )
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(c) $\min(\Delta(N_2))$

Fig. 1. (a) A General FCPN structure $N_1$, where $min(\Delta(N_1)) = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 1)^T\}$. (b) An FCPN structure $N_2$, and (c) The twenty-four elements of $\min(\Delta(N_2))$ generated by the software described in references [22], [24].

(DNF expression) *Compute-DNF* ( (PN Structure) $N$, (controllable transition) $t_c \in T_c$)

Compute $\min(\Delta(N)) = \{\widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_2, \ldots, \widehat{\mathbf{m}}_k\}$ using the software described in reference [24].
$\Theta_{t_c}(N) = \emptyset$
**for** $i = \{1, 2, \ldots, k\}$ **do**
  $Clause_i = \emptyset$ {/* Current Clause */}
  **for** $j = \{1, 2, \ldots, n\}$ **do**
    $\beta = \max\{0, (\widehat{\mathbf{m}}_i(p_j) + \mathbf{IN}_{j,c} - \mathbf{OUT}_{j,c})\}$
    **if** $((\beta > \mathbf{IN}_{j,c})$ && $(Clause_i == \emptyset))$ **then**
      $Clause_i = (\mathbf{m}(p_j) \geq \beta)$
    **end if**
    **if** $((\beta > \mathbf{IN}_{j,c})$ && $(Clause_i\ ! = \emptyset))$ **then**
      $Clause_i \leftarrow Clause_i \wedge (\mathbf{m}(p_j) \geq \beta)$
    **end if**
  **end for**
  **if** $((Clause_i\ ! = \emptyset)$ && $(\Theta_{t_c}(N) == \emptyset))$ **then**
    $\Theta_{t_c}(N) \leftarrow (Clause_i)$
  **end if**
  **if** $((Clause_i\ ! = \emptyset)$ && $(\Theta_{t_c}(N)\ ! = \emptyset))$ **then**
    $\Theta_{t_c}(N) \leftarrow \Theta_{t_c}(N) \vee (Clause_i)$
  **end if**
**end for**
Return $\Theta_{t_c}(N)$

Fig. 2. For a PN structure $N$ where $\Delta(N)$ is right-closed, and $t_c \in T_c$, the $O(nk)$ function *compute-DNF* $(N, t_c)$ returns a DNF formula, $\Theta_{t_c}(N)$, with at most $k$-many clauses, where the literals take the form $(\mathbf{m}(p_i) \geq \beta)$ , where $\beta \in \mathcal{N}^+$.

would have to be control-disabled at some marking in $\Delta(N)$. Using the procedure of figure 2, we associate a *disjunctive-normal-form* (DNF) expression, $\Theta_{t_c}(N)$, with at most *k clauses*, where the *literals* ("$\mathbf{m}(p_i) \geq \beta$") identify if the token-load of a place $p_i$ is greater than or equal to a threshold $\beta$. In this context, $\Theta_{t_c}(N)$ can be viewed as a mapping $\Theta_{t_c}(N) : \mathcal{N}^n \rightarrow \{0, 1\}$, that returns a logical value for each marking. The controllable transition $t_c$ is permitted at marking $\mathbf{m}$ if and only if $\Theta_{t_c}(N)(\mathbf{m}) = 1$. The main result of the paper shows that the control effected using $\Theta_{t_c}(N)$ for each $t_c \in T_c$ that violates the condition of lemma III.1, is equivalent to the minimally restrictive LESP $\mathcal{P}^*_N$.

To motivate the main result, we note that when the procedure of figure 2 is executed on the PN structure $N_1$ of figure 1(a) and the controllable transition $t_1$, we obtain the following DNF expression

$$\Theta_{t_1}(N_1) = (\mathbf{m}(p_1) \geq 3) \vee ((\mathbf{m}(p_4) \geq 1) \wedge (\mathbf{m}(p_5) \geq 1)).$$

Suppose $\mathbf{m}^1$ is a marking such that $t_1 \in T_e(N_1, \mathbf{m}^1)$ and either $(\mathbf{m}^1(p_1) \geq 3)$ or $((\mathbf{m}^1(p_4) \geq 1) \wedge (\mathbf{m}^1(p_5) \geq 1))$, then if $\mathbf{m}^1 \xrightarrow{t_1} \mathbf{m}^2$ in $N_1$, $\mathbf{m}^2 \in \Delta(N_1)$. That is, the controllable transition $t_1$ will be control-enabled under the minimally restrictive LESP $\mathcal{P}^*_{N_1}$ at the marking $\mathbf{m}^1$ also. If $\mathbf{m}^1$ is a marking such that $t_1 \in T_e(N_1, \mathbf{m}^1)$ and $(\mathbf{m}^1(p_1) \leq 2) \wedge ((\mathbf{m}^1(p_4) = 0) \vee (\mathbf{m}^1(p_5) = 0))$, then the minimally restrictive LESP $\mathcal{P}^*_{N_1}$ will not permit the firing of $t_1$ at marking $\mathbf{m}^1$. Therefore, the policy that permits $t_1$ only when the DNF $\Theta_{t_1}(N_1)$ is true is equivalent to the minimally restrictive LESP $\mathcal{P}^*_{N_1}$.

A *β-threshold sensor* at place $p \in \Pi$ produces an output of unity at a marking $\mathbf{m}$ if $\mathbf{m}(p) \geq \beta$; it produces an output of zero otherwise. The structure of the DNF $\Theta_{t_1}(N_1)$, together with the above observation, indicates that 3-threshold sensor at $p_1$, and 1-threshold sensors at $p_4$ and $p_5$ respectively, provides sufficient information for minimally restrictive supervision for liveness in $N_1(\mathbf{m}^0_1)$ for any $\mathbf{m}^0_1 \in \Delta(N_1)$.

The DNF-expression $\Theta_{t_1}(N_2)$ that results when the procedure of figure 2 is executed on the PN structure $N_2$ of figure 1(b) for the controllable transition $t_1$ is shown in figure 3(a). Each of the twenty-four clauses in the DNF-expression $\Theta_{t_1}(N_2)$ corresponds to a member of $\min(\Delta(N_2))$, shown in figure 1(c). It can be seen that if the $i$-th clause is satisfied by a marking $\mathbf{m}^1$ where $t_1 \in T_e(N_2, \mathbf{m}^1)$, and $\mathbf{m}^1 \xrightarrow{t_1} \mathbf{m}^2$, then $\mathbf{m}^2$ is greater than or equal to the $i$-th member of $\min(\Delta(N_2))$. Consequently, if $\Theta_{t_1}(N_2)$ is true for a marking $\mathbf{m}^1 \in \Delta(N_2)$, transition $t_1$ will be control-enabled at the marking $\mathbf{m}^1$ by the minimally restrictive LESP $\mathcal{P}^*_{N_2}$. Conversely, if $t_1$ is control-enabled by $\mathcal{P}^*_{N_2}$ at a marking $\mathbf{m}^1 \in \Delta(N_2)$ and $\mathbf{m}^1 \xrightarrow{t_1} \mathbf{m}^2$, then by the procedure of figure 2, $\exists i \in \{1, 2, \ldots, k\}$ such that $\mathbf{m}^2 \geq \widehat{\mathbf{m}}_i$, and $\Theta_{t_1}(N_2)$ is true for $\mathbf{m}^1$. Similar observations can be made regarding the DNF-expressions $\Theta_{t_2}(N_2)$ and $\Theta_{t_3}(N_2)$ shown in figure 3(b) and 3(c), respectively. The supervisory policy of control-enabling transitions $t_1, t_2$ and $t_3$ in $N_2$ at a marking $\mathbf{m}^1 \in \Delta(N)$ if and only if the DNF-expressions for $\Theta_{t_1}(N_2), \Theta_{t_2}(N_2)$ and $\Theta_{t_3}(N_2)$ are true at marking $\mathbf{m}^1$ is equivalent to the minimally restrictive LESP $\mathcal{P}^*_{N_2}$. This observation is generalized as the main result of this paper, and the following observation is used

```
( (m(p4)>=1) && (m(p5)>=1) ) ||
( (m(p4)>=1) ) ||
( (m(p1)>=3) ) ||
( (m(p1)>=2) ) ||
( (m(p1)>=2) && (m(p4)>=1) ) ||
( (m(p1)>=2) && (m(p5)>=1) ) ||
( (m(p1)>=2) && (m(p7)>=1) ) ||
( (m(p1)>=2) && (m(p8)>=1) ) ||
( (m(p1)>=2) && (m(p9)>=1) ) ||
( (m(p7)>=1) ) ||
( (m(p4)>=1) && (m(p7)>=1) ) ||
( (m(p5)>=1) && (m(p7)>=1) ) ||
( (m(p7)>=2) ) ||
( (m(p7)>=1) && (m(p8)>=1) ) ||
( (m(p7)>=1) && (m(p9)>=1) ) ||
( (m(p8)>=1) ) ||
( (m(p4)>=1) && (m(p8)>=1) ) ||
( (m(p5)>=1) && (m(p8)>=1) ) ||
( (m(p8)>=2) ) ||
( (m(p8)>=1) && (m(p9)>=1) ) ||
( (m(p9)>=1) ) ||
( (m(p4)>=1) && (m(p9)>=1) ) ||
( (m(p5)>=1) && (m(p9)>=1) ) ||
( (m(p9)>=2) )
```

(a) $\Theta_{t_1}(N_2)$

```
( (m(p4)>=1) && (m(p5)>=1) ) ||
( (m(p2)>=2) && (m(p4)>=1) ) ||
( (m(p1)>=2) ) ||
( (m(p1)>=1) && (m(p2)>=2) ) ||
( (m(p1)>=1) && (m(p4)>=1) ) ||
( (m(p1)>=1) && (m(p5)>=1) ) ||
( (m(p1)>=1) && (m(p7)>=1) ) ||
( (m(p1)>=1) && (m(p8)>=1) ) ||
( (m(p1)>=1) && (m(p9)>=1) ) ||
( (m(p2)>=2) && (m(p7)>=1) ) ||
( (m(p4)>=1) && (m(p7)>=1) ) ||
( (m(p5)>=1) && (m(p7)>=1) ) ||
( (m(p7)>=2) ) ||
( (m(p7)>=1) && (m(p8)>=1) ) ||
( (m(p7)>=1) && (m(p9)>=1) ) ||
( (m(p2)>=2) && (m(p8)>=1) ) ||
( (m(p4)>=1) && (m(p8)>=1) ) ||
( (m(p5)>=1) && (m(p8)>=1) ) ||
( (m(p8)>=2) ) ||
( (m(p8)>=1) && (m(p9)>=1) ) ||
( (m(p2)>=2) && (m(p9)>=1) ) ||
( (m(p4)>=1) && (m(p9)>=1) ) ||
( (m(p5)>=1) && (m(p9)>=1) ) ||
( (m(p9)>=2) )
```

(b) $\Theta_{t_2}(N_2)$

```
( (m(p5)>=1) ) ||
( (m(p2)>=1) ) ||
( (m(p1)>=3) ) ||
( (m(p1)>=2) && (m(p2)>=1) ) ||
( (m(p1)>=2) ) ||
( (m(p1)>=2) && (m(p5)>=1) ) ||
( (m(p1)>=2) && (m(p7)>=1) ) ||
( (m(p1)>=2) && (m(p8)>=1) ) ||
( (m(p1)>=2) && (m(p9)>=1) ) ||
( (m(p2)>=1) && (m(p7)>=1) ) ||
( (m(p7)>=1) ) ||
( (m(p5)>=1) && (m(p7)>=1) ) ||
( (m(p7)>=2) ) ||
( (m(p7)>=1) && (m(p8)>=1) ) ||
( (m(p7)>=1) && (m(p9)>=1) ) ||
( (m(p2)>=1) && (m(p8)>=1) ) ||
( (m(p8)>=1) ) ||
( (m(p5)>=1) && (m(p8)>=1) ) ||
( (m(p8)>=2) ) ||
( (m(p8)>=1) && (m(p9)>=1) ) ||
( (m(p2)>=1) && (m(p9)>=1) ) ||
( (m(p9)>=1) ) ||
( (m(p5)>=1) && (m(p9)>=1) ) ||
( (m(p9)>=2) )
```

(c) $\Theta_{t_3}(N_2)$

Fig. 3. (a) The DNF-expression $\Theta_{t_1}(N_2)$ that is obtained when the procedure in figure 2 is executed on the PN structure $N_2$ (cf. figure 1(b)) and transition $t_1$. (b) The DNF-expression $\Theta_{t_2}(N_2)$, and (c) The DNF-expression $\Theta_{t_3}(N_2)$.

in its proof.

**Observation III.2.** ($Clause_i == \emptyset$) *at line 14 of the procedure of figure 2 for the computation of* $\Theta_{t_c}(N)$ *if and only if* $\mathbf{OUT}_{\bullet,c} \geq \widehat{\mathbf{m}}_i$.

*Proof.* (*If*) If $\mathbf{OUT}_{\bullet,c} \geq \widehat{\mathbf{m}}_i$, then $\widehat{\mathbf{m}}_i + \mathbf{IN}_{\bullet,c} - \mathbf{OUT}_{\bullet,c} \leq \mathbf{IN}_{\bullet,c}$. Consequently, for each $j \in \{1, 2, \ldots, n\}$, at line 6 of the procedure of figure 2, $\beta \leq \mathbf{IN}_{j,c}$. In turn, this would imply that after the conclusion of line 13, $Clause_i = \emptyset$.

(*Only If*) If $Clause_i = \emptyset$ at line 14 of the procedure of figure 2, then for all $j \in \{1, 2, \ldots, n\}$, $\beta \leq \mathbf{IN}_{j,c}$ at line 6. Therefore, $\widehat{\mathbf{m}}_i \leq \mathbf{OUT}_{\bullet,c}$. □

To illustrate observation III.2, we note that for the PN structure $N_2$ shown in figure 1(b), $\mathbf{OUT}_{\bullet,6} = (0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0)^T$, which is the fourteenth minimal element of $\Delta(N_2)$ in the list shown in figure 1(c). If we were to run the procedure of figure 2 on $N_2$ for the transition $t_6$, as a consequence of observation III.2, we would have $Clause_{14} = \emptyset$, which is easily verified.

If $\mathbf{OUT}_{\bullet,c} \geq \widehat{\mathbf{m}}_i$, then $\forall \mathbf{m}^1$ such that $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$, $\mathbf{m}^2 \in \Delta(N)$, this in turn implies that the condition of lemma III.1 would be satisfied for $t_c \in T$. Stated in the contrapositive, if the procedure of figure 2 is applied only to those controllable transitions that violate the requirement of lemma III.1, $Clause_i \neq \emptyset$ at line 14, $\forall i \in \{1, 2, \ldots, k\}$.

**Theorem III.3.** *Let* $N = (\Pi, T, \Phi, \Gamma)$ *be a PN structure where*

$$\Delta(N) = \{\mathbf{m}^0 \in \mathcal{N}^{card(\Pi)} \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\}$$

*is right-closed, and* $T = T_c \cup T_u$ $(T_c \cap T_u = \emptyset)$. *For each controllable transition* $t_c \in T_c$ *that does not meet the requirement of lemma III.1, let* $\Theta_{t_c}(N)$ *denote the DNF expression that results when the procedure of figure 2 is applied to the PN structure N and* $t_c$. *Let* $\mathbf{m}^0 \in \Delta(N)$, *the supervisory policy that control-enables a* $t_c \in T_c$ *if and only if* $\Theta_{t_c}(N)$ *is true at a given marking, is equivalent to the minimally restrictive supervisory policy* $\mathcal{P}_N^*$.

*Proof.* As a consequence of the discussion that accompanied observation III.2, the DNF-expression for $\Theta_{t_c}(N)$ does not have any empty clauses.

Suppose $\mathbf{m}^1 \in \Delta(N)$ and $\Theta_{t_c}(N)$ is true at $\mathbf{m}^1$, and $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$. It follows that $\mathbf{m}^1 \geq \mathbf{IN}_{\bullet,c}$, where $\mathbf{IN}_{\bullet,c}$ is the $c$-th column of the input matrix $\mathbf{IN}$.

Suppose $Clause_i$ in the DNF-expression is true at marking $\mathbf{m}^1$.

If there is a literal of the form $(\mathbf{m}(p_j) \geq \beta)$ in $Clause_i$, then $\mathbf{m}^1(p_j) \geq \max\{0, \widehat{\mathbf{m}}_i(p_j) + \mathbf{IN}_{j,c} - \mathbf{OUT}_{j,c}\} (> \mathbf{IN}_{j,c})$, and since $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$, it follows that $\mathbf{m}^2(p_j) > \widehat{\mathbf{m}}_i(p_j)$.

If there is no literal of the form $(\mathbf{m}(p_j) \geq \beta)$ in $Clause_i$, then $\beta = \max\{0, \widehat{\mathbf{m}}_i(p_j) + \mathbf{IN}_{j,c} - \mathbf{OUT}_{j,c}\} \leq \mathbf{IN}_{j,c} \Rightarrow \mathbf{OUT}_{j,c} \geq \widehat{\mathbf{m}}_i(p_j)$. Since $\mathbf{m}^2(p_j) = \mathbf{m}^1(p_j) - \mathbf{IN}_{j,c} + \mathbf{OUT}_{j,c}$, and $\mathbf{m}^1(p_j) \geq \mathbf{IN}_{j,c}$, it follows that $\mathbf{m}^2(p_j) \geq \mathbf{OUT}_{j,c} \geq \widehat{\mathbf{m}}_i(p_j)$.

Therefore, $\mathbf{m}^2 \geq \widehat{\mathbf{m}}_i \Rightarrow \mathbf{m}^2 \in \Delta(N)$. Consequently, $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$ under the supervision of $\mathcal{P}_N^*$ too.

If $\mathbf{m}^1 \xrightarrow{t_c} \mathbf{m}^2$ under the supervision of $\mathcal{P}_N^*$, then $\exists \widehat{\mathbf{m}}_i$ such that $\mathbf{m}^2 \geq \widehat{\mathbf{m}}_i$. Since, $\mathbf{m}^2 = \mathbf{m}^1 - \mathbf{IN}_{\bullet,c} + \mathbf{OUT}_{\bullet,c} \geq \widehat{\mathbf{m}}_i \Rightarrow \mathbf{m}^1 \geq \widehat{\mathbf{m}}_i + \mathbf{IN}_{\bullet,c} - \mathbf{OUT}_{\bullet,c}$, where $\mathbf{OUT}_{\bullet,c}$ is the $c$-th column of the output matrix $\mathbf{OUT}$. Consequently, if there is a literal of the form $(\mathbf{m}(p_j) \geq \beta)$ in $Clause_i$, it will be satisfied by $\mathbf{m}^1$. Therefore, $\Theta_{t_c}(N)$ will be satisfied by $\mathbf{m}^1$. □

The DNF-expression for $\Theta_{t_1}(N_2)$ shown in figure 3(a) can be simplified as $(\mathbf{m}(p_4) \geq 1) \vee (\mathbf{m}(p_1) \geq 2) \vee (\mathbf{m}(p_7) \geq 1) \vee (\mathbf{m}(p_8) \geq 1) \vee (\mathbf{m}(p_9) \geq 1)$. Similarly, $\Theta_{t_3}(N_2)$ shown in figure 3(c) can be simplified as $(\mathbf{m}(p_5) \geq 1) \vee (\mathbf{m}(p_2) \geq 1) \vee (\mathbf{m}(p_1) \geq 2) \vee (\mathbf{m}(p_7) \geq 1) \vee (\mathbf{m}(p_8) \geq 1) \vee (\mathbf{m}(p_9) \geq 1))$. In general, this process can be automated using a variety of tools (for example, the function BooleanMinimize in *Mathematica*).

As a consequence of theorem III.1 we note that $\beta$-threshold sensors at each place $p_j$ identified by the literal "$(\mathbf{m}(p_j) \geq \beta)$" in the simplified expression for $\Theta_{t_c}$ are sufficient for the minimally restrictive LESP $\mathcal{P}_N^*$. Additionally, this DNF-based, minimally restrictive LESP is a static-map that determines the appropriate control-action using just the information collected from these threshold sensors.

One 3-threshold sensor at $p_1$, and two 1-threshold sensors at $p_4$ and $p_5$ are sufficient for the minimally restrictive LESP $\mathcal{P}^*_{N_1}$ for the PN structure $N_1$ shown in figure 1(a). Likewise, the minimally restrictive LESP $\mathcal{P}^*_{N_2}$ can be enforced using a 1-threshold and 2-threshold sensor in the places in the set $\{p_1, p_2, p_7, p_8, p_9\}$, a 1-threshold sensor in the places in the set $\{p_4, p_5\}$ in the PN structure $N_2$ for any $\mathbf{m}^0_2 \in \Delta(N_2)$.

## IV. Concluding Remarks

A *liveness enforcing supervisory policy* (LESP) control-enables the controllable transitions in a PN $N(\mathbf{m}^0)$ in such a way that the supervised PN is live. For a given PN structure $N$, the set

$$\Delta(N) = \{\mathbf{m}^0 \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\}$$

defines the set of initial markings for which there is an LESP. For an initial marking $\mathbf{m}^0 \in \Delta(N)$, the minimally restrictive LESP for $N(\mathbf{m}^0)$, $\mathcal{P}^*_N$, control-disables a controllable transition $t_c \in T_c$ at a marking $\mathbf{m}^1 \in \Delta(N)$ if and only if $\mathbf{m}^1 \stackrel{t_c}{\to} \mathbf{m}^2$ in $N$, and $\mathbf{m}^2 \notin \Delta(N)$.

We restricted our attention to PN structures $N$ where the set $\Delta(N)$ is right-closed. That is, if $\mathbf{m}^1 \in \Delta(N)$ then every $\mathbf{m}^2 \geq \mathbf{m}^1$ is also in $\Delta(N)$. For this class of PN structures we showed that for each controllable transition $t_c \in T_c$, there is a DNF-expression $\Theta_{t_c}(N)$ which is satisfied by a marking $\mathbf{m}$ if and only if the transition $t_c$ is permitted by $\mathcal{P}^*_N$ at the marking $\mathbf{m}$. Therefore, the supervisory policy that permits the firing of a controllable transition $t_c \in T_c$ at a marking $\mathbf{m}$ if and only if $\Theta_{t_c}$ is satisfied by $\mathbf{m}$ is equivalent to the minimally restrictive LESP $\mathcal{P}^*_N$.

Additionally, each literal in $\Theta_{t_c}(N)$ is of the form "$(\mathbf{m}(p_i) \geq \beta)$." As a consequence, we have the observation that $\beta$-*threshold sensors* at selected places provide sufficient information to enforce $\mathcal{P}^*_N$. For a marking $\mathbf{m}$, a $\beta$-threshold sensor at a place $p \in \Pi$ indicates if the token-load of place $p$ is greater than or equal to $\beta$. In order to be robust to sensor-failures, we suggest the use of fault-tolerant techniques like *linear parity checks* (cf. [25], [26], for example) to provide fault-tolerance capabilities to the minimally restrictive LESP that uses the DNF-expressions $\{\Theta_{t_c}\}_{t_c \in T_c}$. The results of this paper could also find use in the synthesis of monitor place constructions for liveness enforcement that are not necessarily *invariant-based*.

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