there is a supervisory policy that enforces liveness in the original PN if and only if there is a similar policy for the equivalent FCPN, a similar result cannot exist for arbitrary, partially controlled PNs.

The existence of a supervisory policy that enforces liveness in an arbitrary PN is undecidable. One could view the observations made in this paper as a reprieve for partially controlled FCPNs in that perhaps there is a computable test for the existence of a supervisory policy that enforces liveness in a partially controlled FCPN. We suggest investigations into this problem as a future research topic.

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Observation 4.3 *Suppose* $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma} \rightarrow \widehat{\mathbf{m}}^j$ *in* $\widehat{N}$ *under the supervision of* $\widehat{\mathcal{P}}$ *and* $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^k$ *in* N *under the supervision of* $\mathcal{P}$*, where* $\mathbf{x}(\widehat{\sigma}) = \mathbf{x}(\sigma)$*. If* $\mathbf{m}^k \rightarrow \overline{\sigma} \rightarrow \mathbf{m}^s$ *in* N *under the supervision of* $\mathcal{P}$*, then* $\widehat{\mathbf{m}}^j \rightarrow \overline{\sigma} \rightarrow \widehat{\mathbf{m}}^r$ *in* $\widehat{N}$ *under the supervision of* $\widehat{\mathcal{P}}$ *also.*

Observation 4.2 can be established by using the definition of $\widehat{\mathcal{P}}$ and an induction argument over the length of $\widehat{\sigma}$. The details are skipped for brevity. Observation 4.3 follows directly from the fact that if $\mathbf{m}^k \rightarrow t_i \rightarrow \mathbf{m}^{k+1}$ under the supervision of $\mathcal{P}$ in N, then $\widehat{\mathbf{m}}^j \rightarrow t_i \rightarrow \widehat{\mathbf{m}}^{j+1}$ under the supervision of $\widehat{\mathcal{P}}$ in $\widehat{N}$. To see this, note that $\widehat{\mathbf{m}}^j \geq \mathbf{m}^k$, which in turn implies $t_i \in T_e(\widehat{\mathbf{m}}^j, \widehat{N})$. So, if $t_i \in T_u$, then $\widehat{\mathbf{m}}^j \rightarrow t_i \rightarrow \widehat{\mathbf{m}}^{j+1}$ under the supervision of $\widehat{\mathcal{P}}$ in $\widehat{N}$. For the case when $t_i \in T_c$, we note that $\mathbf{x}(\widehat{\sigma} t_i) = \mathbf{x}(\sigma t_i)$, and $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma} t_i)_j > \mathbf{x}(\sigma t_i)_j\} = \emptyset (\subseteq T_u)$, which in turn implies that $\widehat{\mathcal{P}}(\widehat{\mathbf{m}}^j)_i = 1$, and $\widehat{\mathbf{m}}^j \rightarrow t_i \rightarrow \widehat{\mathbf{m}}^{j+1}$ under the supervision of $\widehat{\mathcal{P}}$ in $\widehat{N}$. We now present a proof of theorem 4.1.

Proof: We will show that the supervisory policy $\widehat{\mathcal{P}}$ enforces liveness in $\widehat{N}$. Let $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma} \rightarrow \widehat{\mathbf{m}}^j$ under the supervision of $\widehat{\mathcal{P}}$. By observation 4.2 we know that $\exists \sigma \in T^*$, such that (i) $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^k$ under the supervision of $\mathcal{P}$ in N, (ii) $\forall p \in \{1, 2, \ldots, m\}, \mathbf{x}(\widehat{\sigma} t_i)_p \geq \mathbf{x}(\sigma)_p$, and (iii) $\{t_p \in T \mid \mathbf{x}(\widehat{\sigma} t_i)_p > \mathbf{x}(\sigma)_p\} \subseteq T_u$. By observation 4.1 we know that $\exists \overline{\sigma}_1, \overline{\sigma}_2 \in T^*$, such that (i) $\widehat{\mathbf{m}}^j \rightarrow \overline{\sigma}_1 \rightarrow \widehat{\mathbf{m}}^r$ in $\widehat{N}$, (ii) $\mathbf{m}^k \rightarrow \overline{\sigma}_2 \rightarrow \mathbf{m}^s$ under the supervision of $\mathcal{P}$ in N, and (iii) $\mathbf{x}(\widehat{\sigma}\overline{\sigma}_1) = \mathbf{x}(\sigma\overline{\sigma}_2)$ ($\Rightarrow \widehat{\mathbf{m}}^r \geq \mathbf{m}^s$). Additionally, from corollary 4.1 we know that $\forall \overline{\sigma}_3 \in pr(\overline{\sigma}_1), \exists \overline{\sigma}_4 \in pr(\overline{\sigma}_2)$, such that (i) $\widehat{\mathbf{m}}^j \rightarrow \overline{\sigma}_3 \rightarrow \widehat{\overline{\mathbf{m}}}$ in N, (ii) $\mathbf{m}^k \rightarrow \overline{\sigma}_4 \rightarrow \overline{\mathbf{m}}$ under the supervision of $\mathcal{P}$ in N, (iii) $\mathbf{x}(\widehat{\sigma}\overline{\sigma}_3)_p \geq \mathbf{x}(\sigma\overline{\sigma}_4)_p, \forall p \in \{1, \ldots, m\}$, and (iv) $\{t_p \in T \mid \mathbf{x}(\widehat{\sigma}\overline{\sigma}_3)_p > \mathbf{x}(\sigma\overline{\sigma}_4)_p\} \subseteq T_u$. All these observations together imply that $\widehat{\mathbf{m}}^j \rightarrow \overline{\sigma}_1 \rightarrow \widehat{\mathbf{m}}^r$ under the supervision of $\widehat{\mathcal{P}}$ in $\widehat{N}$.

Noting that (i) $\mathbf{x}(\widehat{\sigma}\overline{\sigma}_1) = \mathbf{x}(\sigma\overline{\sigma}_2)$, (ii) $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma}\overline{\sigma}_1 \rightarrow \widehat{\mathbf{m}}^r$ under the supervision of $\widehat{\mathcal{P}}$ in $\widehat{N}$, (iii) $\mathbf{m}^0 \rightarrow \sigma\overline{\sigma}_2 \rightarrow \mathbf{m}^s$ under the supervision of $\mathcal{P}$ in N, and (iv) the fact that $\mathcal{P}$ enforces liveness in N, using observation 4.3, we conclude that $\widehat{\mathcal{P}}$ also enforces liveness in $\widehat{N}$.

♣

In reference [4] it is shown that an arbitrary, completely controlled PN can be converted into a bisimulation-equivalent FCPN and there is a supervisory policy that enforces liveness in the original PN if and only if there is a similar policy for the equivalent FCPN. The result in this paper suggests that a similar result cannot exist for arbitrary, partially controlled PNs. This is because, as pointed out in the introduction section, the existence of a supervisory policy that enforces liveness in an arbitrary, partially controlled PN is not monotone with respect to the initial marking.

However, the existence of a supervisory policy that enforces liveness in an arbitrary, completely controlled PN is monotone with respect to the initial marking. Therefore for the class of completely controlled PNs, the result in reference [4] referred to above, does not contradict the main result in this paper.

5 Conclusions

In this paper we have shown that the existence of a supervisory policy that enforces liveness in a partially controlled FCPN is monotone with respect to the initial marking. That is, if there is a supervisory policy that enforces liveness in a partially controlled FCPN $N = (\Pi, T, \Phi, \mathbf{m}^0)$, then there is a supervisory policy that enforces liveness in the FCPN $\widehat{N} = (\Pi, T, \Phi, \widehat{\mathbf{m}}^0)$, if $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$.

However, this result is not true for the general class of partially controlled PNs. Unlike the case of arbitrary, completely controlled PNs, where it is possible to construct an FCPN, such that

$\widehat{\overline{\mathbf{m}}}_r = 0$, we infer that the term $\mathbf{C}(r, \bullet)\mathbf{y}$ has to be less than zero. Also, if $\mathbf{y}_i > 0$, for some $i \in \{1, \ldots, m\}$, then $t_i \in \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$. This, along with the fact that N (and $\widehat{N}$) is an FCPN, implies that the place p_r is the only input place to one of the transitions in the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$. In turn this would imply that at least one of the transitions in the $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$ set is state-enabled in N under the marking $\overline{\mathbf{m}}$. A contradiction.

In the second case (i.e. $(^\bullet t_q)^\bullet = \{t_q\}$), we note that $\exists p_r \in^\bullet t_q$, such that $\overline{\mathbf{m}}_r \geq 1$, while $\widehat{\overline{\mathbf{m}}}_r = 0$. Using the same logic as in the previous case we conclude that p_r in an input place of one of the transitions in the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$. But the fact that $(^\bullet t_q)^\bullet = \{t_q\}$, $p_r \in^\bullet t_q$ and N (and $\widehat{N}$) are FCPNs, we conclude that the transition t_q is one of the transitions in the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$. A similar contradiction is obtained in this case also.

So, $T_e(\overline{\mathbf{m}}, N) - T_e(\widehat{\overline{\mathbf{m}}}, \widehat{N}) = \emptyset$, and $\widehat{\mathbf{m}}^j \rightarrow \omega_1 \rightarrow \widehat{\mathbf{m}}^{j+1}$ in $\widehat{N}$, while $\mathbf{m}^k \rightarrow \omega_1 \rightarrow \mathbf{m}^{k+1}$ under the supervision of $\mathcal{P}$ in N. At the marking $\mathbf{m}^{k+1}$ in N we can fire one of the enabled transitions $t_s \in \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \cap T_e(\mathbf{m}^{k+1}, N)$. Note that since $t_s \in T_u$, t_s can be fired under the supervision of $\mathcal{P}$ in N at the marking $\mathbf{m}^{k+1}$. Let us suppose $\mathbf{m}^{k+1} \rightarrow t_s \rightarrow \mathbf{m}^{k+2}$ under the supervision of $\mathcal{P}$ in N.

We repeat the above argument replacing the string $\widehat{\sigma}$ with $\widehat{\sigma}\omega_1$, and the string σ with $\sigma\omega_1 t_s$ (or, equivalently, the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\}$ with the set $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} - \{t_s\}$, $\mathbf{m}^k$ with $\mathbf{m}^{k+2}$, and $\widehat{\mathbf{m}}^j$ with $\widehat{\mathbf{m}}^{j+1}$). This process is repeated till the Parikh mapping of the firing sequence in N and $\widehat{N}$ are identical (i.e. there are no uncontrollable transitions that have been fired in $\widehat{N}$ that have not been fired in N).

♣

Corollary 4.1 *For the strings $\overline{\sigma}_1, \overline{\sigma}_2 \in T^*$ referred to in the statement of Observation 4.1, the following observation is also true: $\forall \overline{\sigma}_3 \in pr(\overline{\sigma}_1), \exists \overline{\sigma}_4 \in pr(\overline{\sigma}_2)$, such that (i) $\widehat{\mathbf{m}}^j \rightarrow \overline{\sigma}_3 \rightarrow \widehat{\overline{\mathbf{m}}}$ in N, (ii) $\mathbf{m}^k \rightarrow \overline{\sigma}_4 \rightarrow \overline{\mathbf{m}}$ under the supervision of $\mathcal{P}$ in N, (iii) $\mathbf{x}(\widehat{\sigma}\overline{\sigma}_3)_i \geq \mathbf{x}(\sigma\overline{\sigma}_4)_i, \forall i \in \{1, \ldots, m\}$, and (iv) $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma}\overline{\sigma}_3)_j > \mathbf{x}(\sigma\overline{\sigma}_4)_j\} \subseteq T_u$.*

The above corollary follows directly from the proof of observation 4.1. This can be seen by noting that each prefix of the string ω_1 is a valid firing string starting from $\widehat{\mathbf{m}}^j$ in $\widehat{N}$, and it is also a valid firing string starting from $\mathbf{m}^k$ under the supervision of $\mathcal{P}$. Additionally, the firing of $t_s \in T_u$ at the marking $\mathbf{m}^{k+1}$ does not affect the conditions of the corollary. Using the supervisory policy $\mathcal{P}$ for N, we present a recursive definition of a supervisory policy $\widehat{\mathcal{P}}$ for $\widehat{N}$ as follows:

- $\widehat{\mathcal{P}}(\widehat{\mathbf{m}}^0) = \mathcal{P}(\mathbf{m}^0)$.
- Let us suppose the current marking of the PN $\widehat{N}$ is $\widehat{\mathbf{m}}^j$, where $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma} \rightarrow \widehat{\mathbf{m}}^j$ under the supervision of $\widehat{\mathcal{P}}$,
  - $\forall t_i \in T_u, \widehat{\mathcal{P}}(\widehat{\mathbf{m}}^j)_i = 1$.
  - $\forall t_i \in T_c, \widehat{\mathcal{P}}(\widehat{\mathbf{m}}^j)_i = 1 \Leftrightarrow$ (i) $t_i \in T_e(\widehat{\mathbf{m}}^j, \widehat{N})$, and (ii) $\exists \sigma \in T^*$, such that (a) $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^s$ under the supervision of $\mathcal{P}$ in N, (b) $\forall k \in \{1, 2, \ldots, m\}, \mathbf{x}(\widehat{\sigma} t_i)_k \geq \mathbf{x}(\sigma)_k$, and (c) $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma} t_i)_j > \mathbf{x}(\sigma)_j\} \subseteq T_u$.

The following observations play a critical role in the proof of theorem 4.1.

Observation 4.2 *Suppose $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma} \rightarrow \widehat{\mathbf{m}}^j$ in $\widehat{N}$ under the supervision of $\widehat{\mathcal{P}}$, then $\exists \sigma \in T^*$, such that (i) $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^s$ under the supervision of $\mathcal{P}$ in N, (ii) $\forall k \in \{1, 2, \ldots, m\}, \mathbf{x}(\widehat{\sigma} t_i)_k \geq \mathbf{x}(\sigma)_k$, and (iii) $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma} t_i)_j > \mathbf{x}(\sigma)_j\} \subseteq T_u$.*

3 Review of Relevant Results

Reference [5] contains a necessary and sufficient condition for the existence of a supervisory policy that enforces liveness in an arbitrary PN. This condition is undecidable in general, but if all the transitions in a PN are controllable, or, if the PN is bounded the abovementioned test is decidable. Furthermore, it is possible to synthesize a least-restrictive policy that enforces liveness for these special cases. The computational cost of testing the existence, and synthesizing the least restrictive policy that enforces liveness can be prohibitive in general.

However, if the PN can be represented hierarchically [9], or, if PN has a specific structure [8], the computational cost of testing the existence and synthesis of policies that enforce liveness can be significantly improved. Also, if the PN belongs to a family of PNs [6, 7], there is a ready-made policy that enforces liveness. It is important to note that for these families of PNs the test for the existence and synthesis of policies that enforce liveness is entirely unnecessary.

4 Main Result

Theorem 4.1 *If there is a supervisory policy $\mathcal{P}$ that enforces liveness in a FCPN $N = (\Pi, T, \Phi, \mathbf{m}^0)$, where $T = T_u \cup T_c$, and T_u could possibly be non-empty, then there is a supervisory policy $\widehat{\mathcal{P}}$ that enforces liveness in the FCPN $\widehat{N} = (\Pi, T, \Phi, \widehat{\mathbf{m}}^0)$, if $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$.*

Before we present a proof of theorem 4.1, we present an observation on a relationship between a specific subset of the unsupervised behavior of $\widehat{N}$ and that of N under the supervision of $\mathcal{P}$. This observation plays a critical role in the synthesis of a supervisory policy $\widehat{\mathcal{P}}$ that enforces liveness in $\widehat{N}$.

Observation 4.1 *Suppose $\widehat{\mathbf{m}}^0 \rightarrow \widehat{\sigma} \rightarrow \widehat{\mathbf{m}}^j$ in $\widehat{N}$, $\mathbf{m}^0 \rightarrow \sigma \rightarrow \mathbf{m}^k$ under the supervision of $\mathcal{P}$ in N, $\forall i \in \{1, 2, \ldots, m\}, \mathbf{x}(\widehat{\sigma})_i \geq \mathbf{x}(\sigma)_i$, where $\mathbf{x}(\bullet)$ denotes the Parikh-mapping, or score, of the string-argument and $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$. Also, let us suppose that $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \subseteq T_u$, then $\exists \overline{\sigma}_1, \overline{\sigma}_2 \in T^*$, such that (i) $\widehat{\mathbf{m}}^j \rightarrow \overline{\sigma}_1 \rightarrow \widehat{\mathbf{m}}^r$ in $\widehat{N}$, (ii) $\mathbf{m}^k \rightarrow \overline{\sigma}_2 \rightarrow \mathbf{m}^s$ under the supervision of $\mathcal{P}$ in N, and (iii) $\mathbf{x}(\widehat{\sigma}\overline{\sigma}_1) = \mathbf{x}(\sigma\overline{\sigma}_2)$ ($\Rightarrow \widehat{\mathbf{m}}^r \geq \mathbf{m}^s$).*

Proof: Since the supervisory policy $\mathcal{P}$ enforces liveness in N, we can pick a firing sequence $\omega_1 \in T^*$ with the following properties: (i) $\mathbf{m}^k \rightarrow \omega_1 \rightarrow \mathbf{m}^{k+1}$ under the supervision of $\mathcal{P}$ in N, (ii) $\forall \overline{\omega}_1 \in (pr(\omega_1) - \{\omega_1\})$, if $\mathbf{m}^k \rightarrow \overline{\omega}_1 \rightarrow \overline{\mathbf{m}}$, then $T_e(\overline{\mathbf{m}}, N) \cap \{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} = \emptyset$, where $pr(\bullet)$ is used to denote the prefix-set of the string argument, and (iii) $\{t_j \in T \mid \mathbf{x}(\widehat{\sigma})_j > \mathbf{x}(\sigma)_j\} \cap T_e(\mathbf{m}^{k+1}, N) \neq \emptyset$.

We will show that $\widehat{\mathbf{m}}^j \rightarrow \omega_1 \rightarrow \widehat{\mathbf{m}}^{j+1}$ in $\widehat{N}$ also. This can be established by contradiction. Let us suppose $\exists \overline{\omega}_1 \in (pr(\omega_1) - \{\omega_1\})$, such that $\mathbf{m}^k \rightarrow \overline{\omega}_1 \rightarrow \overline{\mathbf{m}}$ under the supervision of $\mathcal{P}$ in N, $\widehat{\mathbf{m}}^j \rightarrow \overline{\omega}_1 \rightarrow \widehat{\overline{\mathbf{m}}}$ in $\widehat{N}$, and $\exists t_q \in T$, such that $t_q \in T_e(\overline{\mathbf{m}}, N) - T_e(\widehat{\overline{\mathbf{m}}}, \widehat{N})$. We consider two cases (i) $({}^\bullet t_q)^\bullet \neq \{t_q\}$ and (ii) $({}^\bullet t_q)^\bullet = \{t_q\}$.

In the first case, by the fact that N (and $\widehat{N}$) are FCPNs, we infer that the input-place set of t_q is a singleton. Let $\{p_r\} = {}^\bullet t_q$. Since $t_q \in T_e(\overline{\mathbf{m}}, N) - T_e(\widehat{\overline{\mathbf{m}}}, \widehat{N})$, we note $\overline{\mathbf{m}}_r \geq 1$, while $\widehat{\overline{\mathbf{m}}}_r = 0$. But, $\overline{\mathbf{m}}_r = \mathbf{m}^0_r + \mathbf{C}(r, \bullet)\mathbf{x}(\sigma) + \mathbf{C}(r, \bullet)\mathbf{x}(\overline{\omega}_1)$, while $\widehat{\overline{\mathbf{m}}}_r = \widehat{\mathbf{m}}^0_r + \mathbf{C}(r, \bullet)\mathbf{x}(\widehat{\sigma}) + \mathbf{C}(r, \bullet)\mathbf{x}(\overline{\omega}_1)$. The term $\mathbf{C}(r, \bullet)$ is used to denote the r-th row of the incidence matrix, the term $\mathbf{x}(\bullet)$ denotes the Parikh mapping, or score, of the string argument. Since $\forall i \in \{1, \ldots, m\}, \mathbf{x}(\widehat{\sigma})_i \geq \mathbf{x}(\sigma)_i, \Rightarrow \exists \mathbf{y} \in \mathcal{N}^m$, such that $\mathbf{C}(r, \bullet)\mathbf{x}(\widehat{\sigma}) = \mathbf{C}(r, \bullet)\mathbf{x}(\sigma) + \mathbf{C}(r, \bullet)\mathbf{y}$. Since $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$, $\overline{\mathbf{m}}_r \geq 1$, and

- for $i \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_{j_i} produces a marking at which the transition $t_{j_{i+1}}$ is enabled.

The set of *reachable markings* from $\mathbf{m}^0$, denoted by $\Re(N, \mathbf{m}^0)$, is the set of markings generated by all valid firing strings starting with marking $\mathbf{m}^0$ in the PN N. At a marking $\mathbf{m}^1$, if the firing of a valid firing string σ results in a marking $\mathbf{m}^2$, we represent it as $\mathbf{m}^1 \rightarrow \sigma \rightarrow \mathbf{m}^2$. A transition $t \in T$ is *live* if $\forall \mathbf{m}^1 \in \Re(N, \mathbf{m}^0), \exists$ a $\mathbf{m}^2 \in \Re(N, \mathbf{m}^1)$ such that $t \in T_e(\mathbf{m}^2, N)$.

A PN $N = (\Pi, T, \Phi, \mathbf{m}^0)$ is a *Free-Choice* PN (FCPN) if $\forall p \in \Pi, card(p^\bullet) > 1 \Rightarrow^\bullet (p^\bullet) = \{p\}$. In words, a PN is Free-Choice if and only if an arc from a place to a transition is either the unique output arc from that place, or, is the unique input arc to the transition.

In the context of the marking being represented as a nonnegative, integral vector, the i,j-th entry of the $n \times m$ *incidence matrix* $\mathbf{C}$ of the PN N is a matrix defined as

$$\mathbf{C}_{i,j} = \begin{cases} -1 & \text{if } p_i \in {}^\bullet t, \\ 1 & \text{if } p_i \in t^\bullet, \\ 0 & \text{otherwise.} \end{cases}$$

So, if $\mathbf{x}(\sigma)$ is the *Parikh mapping*, or *score*, of any valid firing sequence $\sigma \in T^*$ starting at $\mathbf{m}^0$, the resulting marking $\mathbf{m}$ can be represented as

$$\mathbf{m} = \mathbf{m}^0 + \mathbf{C}\mathbf{x}(\sigma). \tag{2}$$

Given two integer-valued vectors $\mathbf{x}, \mathbf{y}$, we use the notation $\mathbf{x} \geq \mathbf{y}$ if some component of $\mathbf{x}$ is greater than the corresponding component of $\mathbf{y}$ and no component of $\mathbf{x}$ is less than the corresponding component of $\mathbf{y}$.

For the purposes of supervisory control the set of transitions $T = \{t_1, t_2, \ldots, t_m\}$ is partitioned into two sets $T_u = \{t_1, t_2, \ldots, t_p\}$ and $T_c = \{t_{p+1}, t_{p+2}, \ldots, t_m\}$. The set of transitions T_c (T_u) is referred to as the set of *controllable* (*uncontrollable*) transitions. A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \rightarrow \{0, 1\}^m$, is a total map that returns an m-dimensional binary vector for each reachable marking. For reasons that should be obvious in the following paragraph, the supervisory policy must also satisfy the requirement that $\mathcal{P}(\mathbf{m})_i = 1, 1 \leq i \leq p, \forall \mathbf{m} \in \mathcal{N}^n$.

The supervisory policy $\mathcal{P}$ permits the firing of transition t_i at marking $\mathbf{m}$, only if $\mathcal{P}(\mathbf{m})_i = 1$. If at a marking $\mathbf{m}$ all input places to a transition t_i contain at least one token, we say the transition t_i is *state-enabled* at $\mathbf{m}$. If $\mathcal{P}(\mathbf{m})_i = 1$, we say the transition t_i is *control-enabled* at $\mathbf{m}$. A transition has to be state-enabled and control-enabled before it can fire. Since uncontrollable transitions can never be prevented from firing, they remain control-enabled under all markings. Hence the requirement that $\mathcal{P}(\mathbf{m})_i = 1, 1 \leq i \leq p, \forall \mathbf{m} \in \mathcal{N}^n$.

A string of transitions $\sigma = t_{j_1} t_{j_2} \cdots t_{j_k}$, where $t_{j_i} \in T$ ($i \in \{1, 2, \ldots, k\}$) is said to be a *valid firing string* under the supervision of $\mathcal{P}$, starting from the marking $\mathbf{m}$, if,

- the transition t_{j_1} is enabled at the marking $\mathbf{m}$, $\mathcal{P}(\mathbf{m})_{j_1} = 1$, and
- for $i \in \{1, 2, \ldots, k-1\}$ the firing of the transition t_{j_i} produces a marking $\mathbf{m}^i$ at which the transition $t_{j_{i+1}}$ is enabled and $\mathcal{P}(\mathbf{m}^i)_{j_{i+1}} = 1$.

The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$. A transition t_{j_i} is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^1 \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^2 \in \Re(N, \mathbf{m}^1, \mathcal{P})$ such that $t_{j_i} \in T_e(\mathbf{m}^2)$ and $\mathcal{P}(\mathbf{m}^2)_{j_i} = 1$. A supervisory policy $\mathcal{P}$ enforces liveness if all transitions in N are live under $\mathcal{P}$.

In the next section we present a brief overview of some results in the literature on the synthesis of supervisory policies that enforce liveness in PNs.

Petri nets (PNs) [2, 3] are an ideal choice of the modeling of such systems as they allow easy representation of the logical preconditions as the marking, and the operations can be represented as transitions. A PN is said to be *live*, if from any reachable marking it is possible to fire any transition, although not necessarily immediately. A *Free-Choice Petri net* (FCPN) is a restricted class of Petri nets (PNs) where every arc from a place to a transition is either the unique output arc from that place, or, it is the unique input arc to the transition. In this paper we consider *partially controlled* FCPNs where the external agent, the supervisor, can prevent only some (i.e. not all) transitions from firing. We concern ourselves with the synthesis of supervisory policies that enforce liveness in non-live, partially controlled FCPNs. In this paper we show that the existence of a supervisory policy that enforces liveness in partially controlled FCPNs is monotone with respect to the initial marking. That is, if there is a supervisory policy that enforces liveness in a partially controlled FCPN for a given initial marking $\mathbf{m}^0$, then there exists a supervisory policy that enforces liveness in the same FCPN for any initial marking that is larger than $\mathbf{m}^0$.

To check if the same property is true of general PNs, we first note that there is a supervisory policy that enforces liveness in a PN with no controllable transitions if and only if the PN is live to begin with. Additionally, since the property of liveness is not monotone with respect to the initial marking for an arbitrary net (cf. figure 4, reference [1] for an illustrative example), we conclude the main result of this paper does not hold for partially controlled PNs in general. This observation along with the main result of this paper has several implications regarding the relationship between the family of partially controlled FCPNs and partially controlled PNs vis a vis the existence and synthesis of supervisory policies that enforce liveness. We discuss some of these implications in this paper.

The following section contains the notations and definitions used in this paper. Section 4 contains the main result of this paper and some of its implications. The paper concludes with some future research directions in section 5.

2 Notations and Definitions

A *Petri net* (PN) $N = (\Pi, T, \Phi, \mathbf{m}^0)$ is an ordered 4-tuple, where $\Pi = \{p_1, p_2, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, t_2, \ldots, t_m\}$ is a set of m *transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, $\mathbf{m}^0 : \Pi \rightarrow \mathcal{N}$ is the *initial marking function* (or the *initial marking*), and $\mathcal{N}$ is the set of nonnegative integers. The *state* of a PN is the marking $\mathbf{m} : \Pi \rightarrow \mathcal{N}$ that identifies the number of *tokens* in each place. A marking $\mathbf{m} : \Pi \rightarrow \mathcal{N}$ is sometimes represented by an integer-valued vector $\mathbf{m} \in \mathcal{N}^n$, where the i-th component $\mathbf{m}_i$ represents the token load $(\mathbf{m}(p_i))$ of the i-th place. The context should suggest the appropriate usage. For a given marking $\mathbf{m}$ a transition $t \in T$ is said to be *enabled* if $\forall p \in^\bullet t$, $\mathbf{m}(p) \geq 1$, where ${}^\bullet x = \{y \mid (y, x) \in \Phi\}$. The set of enabled transitions in the PN N at the marking $\mathbf{m}$ is denoted by the symbol $T_e(\mathbf{m}, N)$. An enabled transition $t \in T_e(\mathbf{m}^1, N)$ can *fire*, which changes the marking $\mathbf{m}^1$ to $\mathbf{m}^2$ according to the equation

$$\mathbf{m}^2(p) = \mathbf{m}^1(p) - card(p^\bullet \cap \{t\}) + card({}^\bullet p \cap \{t\}), \tag{1}$$

where $x^\bullet = \{y \mid (x, y) \in \Phi\}$, and the symbol $card(\bullet)$ is used to denote the cardinality of the set argument. This notation is also used to denote the predecessor or successor set of a set of places or transitions.

A string of transitions $\sigma = t_{j_1} t_{j_2} \cdots t_{j_k}$, where $t_{j_i} \in T$ $(i \in \{1, 2, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}$, if,

- the transition t_{j_1} is enabled at the marking $\mathbf{m}$, and

On Partially Controlled Free Choice Petri Nets

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Abstract

A *Petri Net* (PN) is said to be *live* if it is possible to fire any transition from every reachable marking, although not necessarily immediately. Under appropriate conditions, a non-live PN can be made live via supervision. Under this paradigm an external-agent, the supervisor, prevents the firing of certain transitions at each reachable marking so as to enforce liveness. A PN is *partially controlled* if the supervisor can prevent the firing of only a subset of transitions. A *Free-Choice Petri net* (FCPN) is a PN where every arc from a place to a transition is either the unique output arc from that place, or, it is the unique input arc to the transition. In this paper we show the existence of a supervisory policy that enforces liveness in a partially controlled FCPN is monotone with respect to the initial marking. That is, if there is a supervisory policy that enforces liveness in a partially controlled FCPN for a particular initial marking, then there is a supervisory policy that enforces liveness for the same FCPN with a larger initial marking. Since this property is not true of the general class of partially controlled PNs, this result has several implications regarding the relationship between the class of partially controlled PNs and partially controlled FCPNs. Some implications of this result along with future research directions are also presented in this paper.

1 Introduction

A large class of systems can be modeled as systems with independent, interacting, concurrent components. Typically, each independent process is split into several operations; the execution of each operation is conditioned on the satisfaction of a set of logical preconditions. Upon the execution of any such operation, a new set of logical conditions is created that inhibit the execution of some operations and enables the execution of others in the system. The supervisory control of such systems requires an external agent to regulate, or limit, the operations of each component so as to guarantee a common objective. In this paper we concern ourselves with a stronger version of deadlock avoidance called *liveness*. From any reachable state of a *live* system, it should be possible for any of the components to execute any of its operations, although not necessarily immediately.

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