Software-Based Synthesis of Maximally Permissive Liveness Enforcing Supervisory Policies for a Class of General Petri Nets

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***Abstract*—A *Petri net* (PN) $N(\mathbf{m}^0)$ is *live* if all of its transitions are potentially fireable from every reachable marking. A PN that is not live can be made live by a *liveness enforcing supervisory policy* (LESP), which decides the set of transitions that are to be permitted at any given marking, such that the supervised-PN is live.**

We assume there are *uncontrollable* transitions that cannot be prevented by the LESP. An LESP is said to be *maximally permissive*, if the fact that it prevents the firing of a transition at a given marking, is sufficient to conclude that all other LESPs would prevent the firing of the same transition at the marking. If there is an LESP for a PN, there is a unique maximally permissive LESP.

This paper is about the synthesis of the maximally permissive LESP for a class of PN models with the help of software tools. The paper concludes with a description of ongoing software development activities.

I. Introduction

We concern ourselves with the problem of synthesizing *liveness enforcing supervisory policies* (LESPs) for a *Petri net* (PN) model of a *Discrete-Event/Discrete-State* (DEDS) system. Essentially, an LESP ensures the *liveness property* [1] that irrespective of the past, all events/activities of a DEDS system can occur at some instant in the future. In a PN model of a DEDS system, this translates to the requirement that every transition in the PN model is potentially fireable for all markings that can be reached under the supervision of the LESP. The DEDS system that is modeled by the PN does not experience *livelocks* or *deadlocks* under the supervision of an LESP, which serves as the primary motivation for this endeavor.

In many instances the LESP can be represented by augmenting the original PN model with extra places, or *monitors*, along with extra arcs between monitors and the existing transitions. The initial token-load of the monitors are determined uniquely by the initial marking of the original PN model. An *invariant-based monitor* ensures the markings of the original PN stays within an appropriately defined polyhedron. The design objective in this case is to find an invariant-based monitor such that the augmented PN is live, and no uncontrollable transition is ever prevented from firing due to insufficient tokens in a monitor at any reachable marking. When these conditions are satisfied, the resulting augmented structure can be interpreted as an implicit description of an LESP for the PN model.

Using two software tools from the literature [2], [3], we present a collection of examples that illustrate the theoretical underpinnings of LESP synthesis. We present the relevant notations and definitions in the next section.

II. Notations, Definitions and Other Preliminaries

We use $\mathcal{N}$ ($\mathcal{N}^+$) to denote the set of non-negative (positive) integers. For $m, n, k \in \mathcal{N}^+$, the set of $m \times n$ matrices (k-dimensional vectors) of non-negative integers is represented as $\mathcal{N}^{m\times n}$ ($\mathcal{N}^k$). The term $card(\bullet)$ denotes the cardinality of the set argument. A *Petri net structure* $N = (\Pi, T, \Phi, \Gamma)$ is an ordered 4-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of *n places*, $T = \{t_1, \ldots, t_m\}$ is a collection of *m transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, and $\Gamma : \Phi \to \mathcal{N}^+$ is the *weight* associated with each arc. In graphical representations of the PN, the weight of an arc is represented by an integer that is placed

along side the arc. For brevity, we refrain from denoting the weight of those arcs $\phi \in \Phi$ where $\Gamma(\phi) = 1$.

The *initial marking function* (or the *initial marking*) of a PN structure N is a function $\mathbf{m}^0 : \Pi \to \mathcal{N}$, which identifies the number of *tokens* in each place. We will use the term *Petri net* (PN) and the symbol $N(\mathbf{m}^0)$ to denote a PN structure N along with its initial marking $\mathbf{m}^0$. A marking $\mathbf{m} : \Pi \to \mathcal{N}$ is sometimes represented by an integer-valued vector $\mathbf{m} \in \mathcal{N}^n$, where the i-th component $\mathbf{m}_i$ represents the token load ($\mathbf{m}(p_i)$) of the i-th place.

We define the sets ${}^\bullet x := \{y \mid (y, x) \in \Phi\}$ and $x^\bullet := \{y \mid (x, y) \in \Phi\}$. A transition $t \in T$ is said to be *enabled* at a marking $\mathbf{m}^i$ if $\forall p \in {}^\bullet t, \mathbf{m}^i(p) \geq \Gamma((p, t))$. The set of enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$. An enabled transition $t \in T_e(N, \mathbf{m}^i)$ can *fire*, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$, where $\forall p \in \Pi, \mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \Gamma(p, t) + \Gamma(t, p)$.

In contexts where the marking is interpreted as a vector, it is useful to define the *input matrix* **IN** and *output matrix* **OUT** as two $n \times m$ matrices, where $\mathbf{IN}_{i,j} = \Gamma(p_i, t_j)$ (resp. $\mathbf{OUT}_{i,j} = \Gamma(t_j, p_i)$) if and only if $p_i \in {}^\bullet t_j$ (resp. $p_i \in t_j^\bullet$). The *incidence matrix* **C** of the PN N is an $n \times m$ matrix, where $\mathbf{C} = \mathbf{OUT} - \mathbf{IN}$.

A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* [4] if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \geq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 \in \mathcal{M}))$. Every right-closed set of vectors $\mathcal{M} \subseteq \mathcal{N}^n$ contains a finite set of minimal-elements, $min(\mathcal{M}) \subset \mathcal{M}$, which can be used to represent $\mathcal{M}$.

A. Supervisory Control of PNs

We assume there is a subset of *controllable transitions*, denoted by $T_c \subseteq T$, can be prevented from firing by an external agent called the *supervisor*. The set of *uncontrollable transitions* ($T_u \subseteq T$) is given by $T_u = T - T_c$. The controllable (resp. uncontrollable) transitions are represented as filled (resp. unfilled) boxes in representation of PNs.

A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$, is a function that returns a 0 or 1 for each transition and each reachable marking. The supervisory policy $\mathcal{P}$ permits the firing of transition t_j at marking $\mathbf{m}^i$, only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $t_j \in T_e(N, \mathbf{m}^i)$ for some marking $\mathbf{m}^i$, we say the transition t_j is *state-enabled* at $\mathbf{m}^i$. If $\mathcal{P}(\mathbf{m}^i, t_j) = 1$, we say the transition t_j is *control-enabled* at $\mathbf{m}^i$. A transition has to be state- and control-enabled before it can fire. The fact that uncontrollable transitions cannot be prevented from firing by the supervisory policy is captured by the requirement that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$. This is implicitly assumed of any supervisory policy.

A string of transitions $\sigma = t_1 \cdots t_k$, where $t_j \in T (j \in \{1, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$, if, (1) $t_1 \in T_e(N, \mathbf{m}^i), \mathcal{P}(\mathbf{m}^i, t_1) = 1$, and (2) for $j \in \{1, \ldots, k-1\}$ the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{j+1}) = 1$.

The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\mathfrak{R}(N, \mathbf{m}^0, \mathcal{P})$. If $\mathbf{m}^{i+k}$ results from the firing of $\sigma \in T^*$ starting from the initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^{i+k}$. A transition t_k is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \mathfrak{R}(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \mathfrak{R}(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$.

A policy $\mathcal{P}$ is a *liveness enforcing supervisory policy* (LESP) for $N(\mathbf{m}^0)$ if all transitions in $N(\mathbf{m}^0)$ are live under $\mathcal{P}$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if for every LESP $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \to \{0, 1\}$ for $N(\mathbf{m}^0)$, the following condition holds $\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t)$.

1) Invariant Based Monitors: The PN structure N can be augmented with the addition of extra places $\Pi_c = \{c_1, \ldots, c_k\}$ ($\Pi \cap \Pi_c = \emptyset$), or *monitors*, along with extra arcs $\Phi_c \subseteq (\Pi_c \times T) \cup (T \times \Pi_c)$ and their associated weights $\widehat{\Gamma} : \Phi_c \to \mathcal{N}^+$, to form a new structure $N_c = (\Pi \cup \Pi_c, T, \Phi \cup \Phi_c, \Gamma_c)$, where

$$\Gamma_c(\phi) = \begin{cases} \Gamma(\phi) & \text{if } \phi \in \Phi, \\ \widehat{\Gamma}(\phi) & \text{if } \phi \in \Phi_c. \end{cases}$$

Following [5], when we deal with markings of N_c as $(n + k)$-dimensional vectors, we suppose the members of the place set of N_c are ordered as follows $\{p_1, \ldots, p_n, c_1, \ldots, c_k\}$, where $\Pi = \{p_1, \ldots, p_n\}$ and $\Pi_c = \{c_1, \ldots, c_k\}$. The initial token load of the monitors in Π_c are determined from the initial marking $\mathbf{m}^0$, according to $\Theta : \mathcal{N}^n \to \mathcal{N}^k$. The PN structure N_c with an initial marking of $((\mathbf{m}^0)^T \ \Theta(\mathbf{m}^0)^T)^T$ is represented as $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$. The set of markings that can be reached from the initial marking $((\mathbf{m}^0)^T \ \Theta(\mathbf{m}^0)^T)^T$ in N_c is denoted by $\mathfrak{R}(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$. Following the aforementioned convention, each $\mathbf{m} \in \mathfrak{R}(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$ can be interpreted as $\mathbf{m} = (\mathbf{m}_1^T \ \mathbf{m}_2^T)^T$, where the vector $\mathbf{m}_1 \in \mathcal{N}^n$ ($\mathbf{m}_2 \in \mathcal{N}^k$) corresponds to the token load of places in Π (Π_c).

Since there might be arcs in Φ_c that originate from some $c_i \in \Pi_c$ to some uncontrollable transition $t_u \in T_u$, we must require $\forall \mathbf{m} \in \mathfrak{R}(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$, $(\forall p \in ({}^\bullet t_u \cap \Pi), \mathbf{m}(p) \geq \Gamma_c((p, t_u))) \Rightarrow (\forall c \in ({}^\bullet t_u \cap \Pi_c), \mathbf{m}(c) \geq \Gamma_c((c, t_u)))$. That is, no uncontrollable transition is prevented from firing at some marking that is reachable in $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$ due to a lack of tokens in the monitors. The requirement, $(\Pi_c \times T_u) \cap \Phi_c = \emptyset$, which supposes that there is no arc from a monitor to an uncontrollable transition in N_c, is sufficient but not necessary, for the above condition to be true [5].

For $\mathbf{A} \in \mathcal{N}^{k \times n}$, $\mathbf{b} \in \mathcal{N}^k$, an initial marking $\mathbf{m}^0 \in \mathcal{N}^n$ where $\mathbf{A}\mathbf{m}^0 \geq \mathbf{b}$, and $\Theta(\mathbf{m}^0) = \mathbf{A}\mathbf{m}^0 - \mathbf{b}$, an *invariant-based* monitor ensures $\forall (\mathbf{m}_1^T \ \mathbf{m}_2^T)^T \in \mathfrak{R}(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$, $\mathbf{A}\mathbf{m}_1 \geq \mathbf{b}$ and $\mathbf{m}_2 = \mathbf{A}\mathbf{m}_1 - \mathbf{b} \geq \mathbf{0}$ (cf. [6], [7]). That is, $\forall (\mathbf{m}_1^T \ \mathbf{m}_2^T)^T \in \mathfrak{R}(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$, the property $\mathbf{A}\mathbf{m}_1 \geq \mathbf{b}$, remains *invariant* for all reachable markings. When applicable, the invariant-based monitor is defined by the monitor incidence-matrix $\mathbf{AC}$, where $\mathbf{C}$ is the incidence matrix of the original PN structure N (cf. section III, [7]).

Liveness enforcement using invariant-based monitors seeks to augment the PN $N(\mathbf{m}^0)$ as described above, such that $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$ is live. When this objective is achieved, the influence of the monitors can be interpreted as an implicit definition of an LESP for the PN $N(\mathbf{m}^0)$.

For any PN structure N, we define the set of initial markings $\mathbf{m}^0$ of N for which there is an LESP for $N(\mathbf{m}^0)$, as follows

$$\Delta(N) = \{\mathbf{m}^0 \in \mathcal{N}^n \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\}.$$

The set $\Delta(N)$ is *control invariant* with respect to N. That is, if $\mathbf{m}^1 \in \Delta(N), t_u \in T_e(N, \mathbf{m}^1) \cap T_u$ and $\mathbf{m}^1 \xrightarrow{t_u} \mathbf{m}^2$ in N, then $\mathbf{m}^2 \in \Delta(N)$. Alternately, only the firing of a controllable transition at a marking in $\Delta(N)$ can result in a new marking that is not in $\Delta(N)$. The PN $N(\mathbf{m}^0)$ has an LESP if and only if $\mathbf{m}^0 \in \Delta(N)$. The supervisory policy that control-disables a transition only if its firing at a marking in $\Delta(N)$ would result in a new marking that is not in $\Delta(N)$, is the minimally restrictive LESP for $N(\mathbf{m}^0)$ for $\mathbf{m}^0 \in \Delta(N)$ (cf. lemma 5.9, [8]).

Reference [5] show that the minimally restrictive LESP $\mathcal{P}$ that ensures $\forall \mathbf{m}^0 \in \Delta(N)$, $\mathfrak{R}(N, \mathbf{m}^0, \mathcal{P}) \subseteq \Delta(N)$, can be represented equivalently by an invariant-based monitor that ensures the liveness of $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$ if and only if $\Delta(N)$ is convex. $\Delta(N)$ is convex if and only if $\min(\Delta(N)) = \min(conv(\Delta(N)) \cap \mathcal{N}^n)$, where $conv(\mathcal{M})$ is the convex hull of the set $\mathcal{M} \subseteq \mathcal{N}^n$.

Unfortunately, the set $\Delta(N)$ is uncomputable for a general PN structure N (cf. corollary 5.2, [9]; theorems 3.1 and 3.2, [8]). However, the set $\Delta_f(N) = \{\mathbf{m}^0 \in \mathcal{N}^n \mid \exists \text{ LESP for } N(\mathbf{m}^0) \text{ if } T_c = T\}$, which is the set of initial markings for which there is an LESP for $N(\mathbf{m}^0)$, assuming all transitions in N are controllable, is an outer-approximation of $\Delta(N)$. That is, $\Delta(N) \subseteq \Delta_f(N)$. The set $\Delta_f(N)$ is right closed for any PN structure N, and can be computed (cf. section IV, [8]), and is used as an initial starting-point to arrive at $\Delta(N)$ on a few instances.

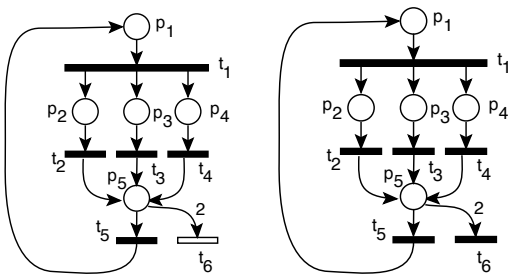
As an illustration, consider the PN structure N_1 shown in figure 1(a). This structure belongs to a class of PNs called *General Free Choice*[1] PNs. The PN structure that would result if all transitions of N_1 are assumed controllable, is shown as PN structure N_2 in figure 1(b). Figure 2 shows the output generated by the software described in reference [2], which presents the minimal elements that define the right-closed set $\Delta(N_2)$. It follows that $\Delta_f(N_1) = \Delta(N_2) = \{\mathbf{m} \in \mathcal{N}^5 \mid \mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_4) + \mathbf{m}(p_5) \geq 1\}$. Additionally, $\Delta(N_1) = \{\mathbf{m} \in \mathcal{N}^5 \mid (\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_3) \geq 1) \vee (\mathbf{m}(p_5)_{mod2} = 1)\}$. It is not hard to see that $\Delta_f(N_1)(= \Delta(N_2))$ is right-closed, but $\Delta(N_1)$ is not right-closed.

The control invariance of $\Delta(N_1)$ with respect to N_1 follows from the fact that each firing of the uncontrollable transition t_6 at marking $\mathbf{m}^1 \in \Delta(N_1)$ will take two (i.e. an even number of) tokens out of p_5, if $\mathbf{m}^1 \xrightarrow{t_6} \mathbf{m}^2$ in N_1, then (i) if $\mathbf{m}^1(p_1) + \mathbf{m}^1(p_2) + \mathbf{m}^1(p_3) + \mathbf{m}^1(p_3) \geq 0$, then $\mathbf{m}^2(p_1) + \mathbf{m}^2(p_2) + \mathbf{m}^2(p_3) + \mathbf{m}^2(p_3) \geq 0$, and $\mathbf{m}^2 \in \Delta(N_1)$, or (b) if $\mathbf{m}^1(p_1) + \mathbf{m}^1(p_2) + \mathbf{m}^1(p_3) + \mathbf{m}^1(p_3) = 0$ then it must be that $\mathbf{m}^1(p_5)_{mod2} = 1$, consequently $(\mathbf{m}^2(p_5)_{mod2} = 1)$ and $\mathbf{m}^2 \in \Delta(N_2)$. The minimally restrictive LESP for $N_1(\mathbf{m}_1^0)$, $\mathbf{m}_1^0 \in \Delta(N_1)$ is shown in figure 1(c).

The PN structure N_3 shown in figure 1(d) is essentially the same as that of N_1, save the fact that transitions t_1 and t_5 are uncontrollable (in addition to transition t_6) in N_3. $\Delta(N_3) = \Delta(N_1)$, because if $\mathbf{m}^1 \xrightarrow{t_1} \mathbf{m}^2$ in N_1, and $\mathbf{m}^1 \in \Delta(N_1)$, then $\mathbf{m}^2 \in \Delta(N_1)$ too. Consequently, the minimally restrictive

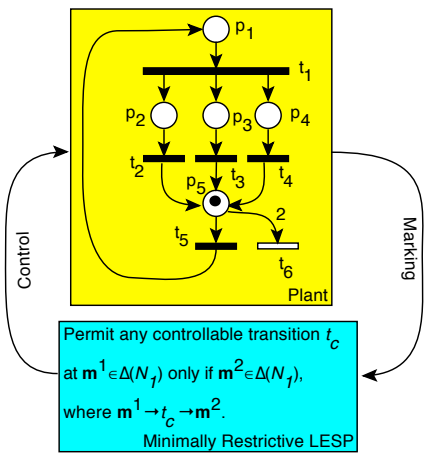
[1] A PN structure $N = (\Pi, T, \Phi, \Gamma)$ is *Free-Choice* (FC) if $\forall p \in \Pi, (card(p^\bullet) > 1 \Rightarrow {}^\bullet(p^\bullet) = \{p\})$, where $card(\bullet)$ denotes the cardinality of the set argument. The PN structure N_1 is a *general* PN as $\Gamma(p_5, t_6) = 2$.

LESP of figure 1(c) would never control-disable t_1; and, it could just as well be an uncontrollable transition, at least as far as the LESP of figure 1(c) is concerned. Likewise, $\mathbf{m}^1 \xrightarrow{t_6} \mathbf{m}^2$ in N_1, and $\mathbf{m}^1 \in \Delta(N_1)$, then $\mathbf{m}^2 \in \Delta(N_1)$ too. Following the same logic as with transition t_1, we conclude that as far as the minimally restrictive LESP of figure 1(c) is concerned, transition t_6 might be considered as an uncontrollable transition as well. The minimally restrictive LESP of figure 1(c) is therefore a minimally restrictive LESP for $N_3(\mathbf{m}_3^0)$ for any $\mathbf{m}_3^0 \in \Delta(N_3)(= \Delta(N_1))$ as well.

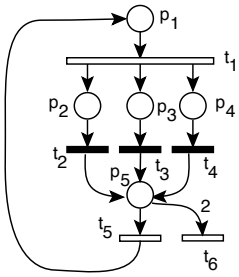


(a) N_1

(b) N_2



(c) Minimally Restrictive LESP



(d) N_3

Fig. 1. *General Free Choice* PN structures (a) $N_1 = (\Pi_1, T_1, \Phi_1, \Gamma_1)$, (a) $N_2 = (\Pi_2, T_2, \Phi_2, \Gamma_2)$, which is the structure N_1, where all transitions are controllable (c) The minimally restrictive LESP for $N_1(\mathbf{m}_1^0)$ for any $\mathbf{m}_1^0 \in \Delta(N_1)$ (d) $N_3 = (\Pi_3, T_3, \Phi_3, \Gamma_3)$, which is the structure N_1 where t_6 and t_1 are uncontrollable (in addition to t_7).

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pn24.lesp        Fri Dec 06 22:12:26 2013        1

 Incidence Matrix :

    T  1  2  3  4  5  6
 P
   1  -1  .  .  .  1  .
   2   1 -1  .  .  .  .
   3   1  . -1  .  .  .
   4   1  .  . -1  .  .
   5   .  1  1  1 -1 -2

Initial Marking: (  1  0  0  0  0)

There is an LESP for this (fully controlled) PN

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The minimal markings that characterize the Minimally Restrictive LESP:
  1: ( 1 0 0 0 0 )
  2: ( 0 1 0 0 0 )
  3: ( 0 0 1 0 0 )
  4: ( 0 0 0 1 0 )
  5: ( 0 0 0 0 1 )
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Fig. 2. The output file generated by the software described in references [2] for the general structure shown in figure 1(b).

The PN structure N_4 shown in figure 3(a) belongs to a class of PNs called *Asymmetric Choice*[2] PNs. $\Delta_f(N_4) = \{\mathbf{m} \in \mathcal{N}^6 \mid \mathbf{m}(p_1)+\mathbf{m}(p_2)+\mathbf{m}(p_3)+\mathbf{m}(p_4)+\mathbf{m}(p_6) \geq 1\}$ (cf. figure 4). The PN structure N_5 of figure 3(c) has t_2 as its only controllable transition.

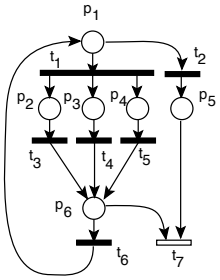
The set $\Delta(N_4) = \Delta_f(N_4) - \{\mathbf{m} \in \mathcal{N}^6 \mid \mathbf{m}(p_5) \geq 3\mathbf{m}(p_1)+\mathbf{m}(p_2)+\mathbf{m}(p_3)+\mathbf{m}(p_4)+\mathbf{m}(p_6)\} = \{\mathbf{m} \in \mathcal{N}^6 \mid 3\mathbf{m}(p_1)+\mathbf{m}(p_2)+\mathbf{m}(p_3)+\mathbf{m}(p_4)+\mathbf{m}(p_6)-\mathbf{m}(p_5) \geq 1\}$. The control invariance of $\Delta(N_4)$ follows from the fact that the firing of the uncontrollable transition t_7 at any marking in $\Delta(N_4)$ results in a new marking that is also in $\Delta(N_4)$. Additionally, $\Delta(N_4)$ is not right-closed. Using a similar logic as with N_1 and N_3, we can establish that $\Delta(N_5) = \Delta(N_4)$, and the minimally restrictive LESP for $N_5(\mathbf{m}_5^0)$ for any $\mathbf{m}_5^0 \in \Delta(N_5)$ is shown in figure 3(c). Since $\Delta(N_5)(= \Delta(N_4))$ is convex, it follows that there is an invariance-based monitor that is equivalent to the minimally restrictive LESP of figure 3(c) (cf. [5]), and it is shown in figure 3(d).

There is a large class of PN families for which the set $\Delta(N)$ is known to be right-closed (cf. [8], [11], [12], [13]), and the software of reference [2] can compute $\Delta(N)$ for any N from these families.

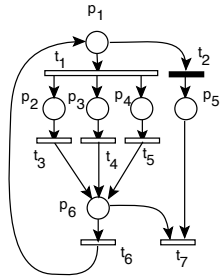
The general PN structure N_6 shown in figure 5(a), which is a member of class $\mathcal{F}$, identified in [12]. Consequently, $\Delta(N_6)$ is right-closed and its minimal elements can be computed using the software described in reference [2]. The input file that describes the PN $N_6(\mathbf{m}_6^0)$ is shown in figure 5(b). Figure 6 shows the five minimal elements of $\min(\Delta(N_6))$.

To compute the convex hull of $\Delta(N_6)$, each member of $\min(\Delta(N_6))$ is elevated by (the four) unit-vectors. This creates a set of 20 new integral vectors.

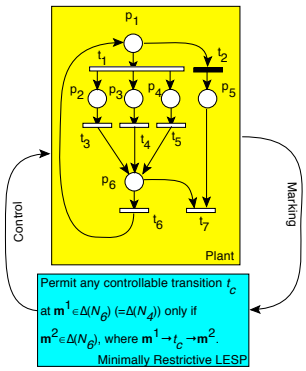
[2]cf. (page 554, [10]) for a formal definition.



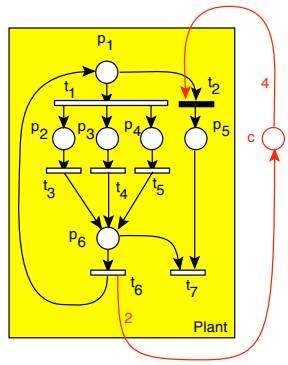
(a) N_4



(b) N_5



(c) Minimally Restrictive LESP



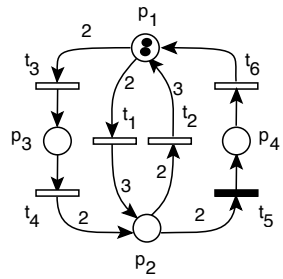
(d) Invariance-Based Monitor

Fig. 3. *Asymmetric Choice* PN structures (a) $N_4 = (\Pi_4, T_4, \Phi_4, \Gamma_4)$, (b) $N_5 = (\Pi_5, T_5, \Phi_5, \Gamma_5)$, which is essentially the structure N_4, where t_2 is the only controllable transition (c) The minimally restrictive LESP for $N_5(\mathbf{m}_5^0)$ for any $\mathbf{m}_5^0 \in \Delta(N_5)(= \Delta(N_4))$ (e) An invariance-based monitor that is equivalent to the minimally restrictive LESP of figure 3(c). The initial token load of the monitor place c is given by the expression $3\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_4) + \mathbf{m}(p_6) - \mathbf{m}(p_5) - 1$.

```
pn23.lesp        Fri Dec 06 17:39:09 2013        1

 Incidence Matrix :

    T  1  2  3  4  5  6  7
 P
   1  -1 -1  .  .  .  1  .
   2   1  . -1  .  .  .  .
   3   1  .  . -1  .  .  .
   4   1  .  .  . -1  .  .
   5   .  1  .  .  .  . -1
   6   .  .  1  1  1 -1 -1

Initial Marking: (  1  0  0  0  0  0)

There is an LESP for this (fully controlled) PN

-----------------------------------------------------

The minimal markings that characterize the Minimally Restrictive LESP:
  1: ( 1 0 0 0 0 0 )
  2: ( 0 1 0 0 0 0 )
  3: ( 0 0 1 0 0 0 )
  4: ( 0 0 0 1 0 0 )
  5: ( 0 0 0 0 0 1 )
```

Fig. 4. The output file generated by the software described in references [2] for $\Delta_f(N_4)$.

These new markings, together with the 5 members of $\min(\Delta(N_6))$ form a set of 25 integral vectors. The polytope that is constructed with these 25 integral vectors has 27 integral points, and has the following



(a) $N_6(\mathbf{m}_6^0)$

```
4 6
2 0 2 0 0 0
0 2 0 0 2 0
0 0 0 1 0 0
0 0 0 0 0 1
0 3 0 0 0 1
3 0 0 2 0 0
0 0 1 0 0 0
0 0 0 0 1 0
2 0 0 0
0 0 0 0 1 0
```

(b) Input File

Fig. 5. (a) The general PN $N_6(\mathbf{m}_6^0)$, where $N_6 \in \mathcal{F}$ ($\Rightarrow \Delta(N_6)$ is right-closed). (b) The input file for the software described in reference [2] that describes $N_6(\mathbf{m}_6^0)$. There are $n = 4$ places and $m = 6$ transitions in this PN, which is represented by the first line. This is followed by the $n \times m$ **IN** and $n \times m$ **OUT** matrices. The n-long line that follows these matrices represents the initial marking $\mathbf{m}_6^0$, which places two tokens in p_1. The m-long line in the end of the file indicates, through a sequence of 1's (0's) the controllable (uncontrollable) transitions in N_6. t_5 is only controllable transition.

```
Input File = "pn1"

Incidence Matrix :

   T  1  2  3  4  5  6
P
  1  -2  3 -2  .  .  1
  2   3 -2  .  2 -2  .
  3   .  .  1 -1  .  .
  4   .  .  .  .  1 -1

Initial Marking : ( 2 0 0 0 )

There is an LESP for this (fully controlled) PN

_____________________________________________


Minimal Elements of the fully controlled Net
--------------------------------------------

 1: ( 0 0 1 0 )
 2: ( 1 0 0 1 )
 3: ( 0 0 0 2 )
 4: ( 0 2 0 0 )
 5: ( 2 0 0 0 )


List of Controllable Transitions
--------------------------------
t5


(Final) Minimal Elements of the control-invariant set
-----------------------------------------------------

 1: ( 0 0 1 0 )
 2: ( 1 0 0 1 )
 3: ( 0 0 0 2 )
 4: ( 0 2 0 0 )
 5: ( 2 0 0 0 )

This is An LESP
```

Fig. 6. The output file generated from the input file of figure 5(b).

facets (cf. figure 7):

$$
\begin{aligned}
-2 + \mathbf{m}(p_1) + \mathbf{m}(p_2) + 2\mathbf{m}(p_3) + \mathbf{m}(p_4) &\geq 0 \quad (1)\\
\mathbf{m}(p_1) &\geq 0 \quad (2)\\
4 - \mathbf{m}(p_1) - \mathbf{m}(p_2) - 2\mathbf{m}(p_3) - \mathbf{m}(p_4) &\geq 0 \quad (3)\\
\mathbf{m}(p_2) &\geq 0 \quad (4)\\
\mathbf{m}(p_3) &\geq 0 \quad (5)\\
\mathbf{m}(p_4) &\geq 0 \quad (6)\\
3 - \mathbf{m}(p_1) - \mathbf{m}(p_2) - \mathbf{m}(p_3) - \mathbf{m}(p_4) &\geq 0 \quad (7)
\end{aligned}
$$

Of these, we can ignore the facets are not right-closed (i.e. equations 3 and 7). Since markings are always non-negative, we can drop equations 2, 4, 5, 6, as they are implicitly true. This means, $conv(\Delta(N_6))$ is represented by equation 1.

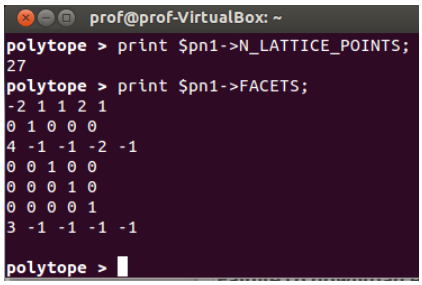


Fig. 7. Polymake output that shows there are 27 integral vectors in the polytope identified by the 25 integral vectors constructed from $\min(\Delta(N_6))$. The half-plane representation of the facets of this polytope are listed above.

The minimal elements of set of markings that satisfy equation 1, $\min(conv(\Delta(N_6)))$, are $\{(0\ 0\ 0\ 2), (0\ 0\ 1\ 0), (0\ 1\ 0\ 1), (0\ 2\ 0\ 0), (1\ 0\ 0\ 1), (1\ 1\ 0\ 0), (2\ 0\ 0\ 0)\}$ ($\neq \min(\Delta(N_6))$). Consequently, $\Delta(N_6)$ is not convex, which can be verified as follows

$$\underbrace{(0\ 1\ 0\ 1)}_{\in conv(\Delta(N_6))-\Delta(N_6)} = \frac{1}{2} \times \underbrace{(0\ 2\ 0\ 0)}_{\in\Delta(N)} + \frac{1}{2} \times \underbrace{(0\ 0\ 0\ 2)}_{\in\Delta(N_6)},$$

and from the results in [5], there is no invariant-based monitor that is equivalent to the minimally restrictive LESP that is based on $\Delta(N_6)$.

III. Conclusion

In this paper we presented a series of examples that illustrate and clarify the theoretical concepts in references [8], [11], [12], [5], [13]. This expository treatment via examples was possible with the help of the software described in references [2]. Our ongoing software development activities involve the synthesis of invariant-based monitors that are equivalent to LESPs that use right-closed sets of markings. Additionally, we are exploring automating "divide-and-conquer" approaches (cf. [14], [15], for example) to the software that computes the members of $\min(\Delta(N))$ when N belongs to a family of PN structures where $\Delta(N)$ is known to be right-closed.

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