On Invariant-Based Monitors that Enforce Liveness in a Class of Partially Controlled General Petri Nets

E. Salimi* and R. S. Sreenivas†*, *Senior Member,*
* ISE & CSL, University of Illinois at Urbana-Champaign, Urbana, IL 61820.
†Corresponding Author.

***Abstract*—We consider a class of *Petri Net* structures where the existence of a *liveness enforcing supervisory policy* (LESP) for an initial marking implies there is a LESP for a larger initial marking. That is, the set of initial markings for which there is a LESP for any instance of this class is *right-closed*. If a transition is prevented from firing at a marking by a LESP, and *all* LESPs, irrespective of the implementation-paradigm that is chosen, prescribe the same control for the marking, then it is a *minimally restrictive* LESP. It is possible to synthesize the minimally restrictive LESP for any instance of this class that uses this right-closed set of markings. Alternately, one could consider *invariant-based monitors* for liveness enforcement in an instance of this class.**

If and when they exist, invariant-based monitors that enforce liveness are *not* minimally restrictive, in general. In this paper, we derive a necessary and sufficient condition for the existence of a minimally restrictive, invariant-based monitor for the class of PNs introduced above.

***Index Terms*—Supervisory Control, Discrete Event Systems, Petri Nets, Liveness.**

I. Introduction

In this paper we consider the problem of synthesizing *liveness enforcing supervisory policies* (LESPs) in *Petri Net* (PN) models of *Discrete-Event/Discrete-State* (DEDS) systems. Events are represented by *transitions*, and the state of the DEDS system is described by the *marking* of the PN model. Some events that are external to the system are *uncontrollable*, in that their execution cannot be prevented by a supervisory policy. These events are represented as uncontrollable transitions in the PN model. The remaining transitions of the PN are *controllable*, and a LESP determines which controllable transition is to be prevented from firing at any giving marking, such that the supervised PN is *live*. Additionally, it is of interest to ensure the LESP is *minimally restrictive*. That is, if a minimally restrictive LESP prevents the firing of an enabled transition at some marking, then *all* LESPs should prescribe the same action for the same marking.

For any PN structure N, the minimally restrictive LESP can be characterized using the set, $\Delta(N)$, which is the set of initial markings of N for which there is a LESP. The set $\Delta(N)$ is *control invariant* (cf. [1]) with respect to N. That is, the firing of any uncontrollable transition at any marking in $\Delta(N)$ will always result in a new marking that is in $\Delta(N)$. When the initial marking $\mathbf{m}^0 \in \Delta(N)$, the minimally restrictive LESP for the PN $N(\mathbf{m}^0)$ disables a controllable transition in N at a marking in $\Delta(N)$ if its firing would result in a new marking that is not in $\Delta(N)$. Consequently, the minimally restrictive LESP is implicitly defined when the set $\Delta(N)$ is known.

In this paper, we consider the class of PN structures where the set $\Delta(N)$ is *right-closed*. That is, if a marking is in $\Delta(N)$, all markings that are (term-wise) larger than it are also in $\Delta(N)$. For instance, if the fact that a transition t in a PN structure N is enabled at a marking is sufficient to infer that all uncontrollable transitions that share a common input place with t are also enabled at the same marking, then $\Delta(N)$ is right-closed [2]. This is case, when N belongs to the family of ordinary *Free-Choice* PNs (FCPNs) [3], and some families of general PNs (cf. [4], [5], [2]). The set $\Delta(N)$ is identified by its minimal elements $\min(\Delta(N))$, which can be computed using the software tool described in reference [6].

In many instances the LESP can be represented by augmenting the original PN model with extra places, or *monitors*, along with extra arcs between monitors and the existing transitions. The initial token-load of the monitors are determined uniquely by the initial marking of the original PN model. An *invariant-based monitor* is an instance of monitor based supervision that ensures the markings of the original PN stay within an appropriately defined polyhedron.

The main result of this paper is that for a PN $N(\mathbf{m}^0)$ with a right-closed $\Delta(N)$-set, there is a minimally restrictive, invariant-based monitor that enforces liveness if and only if $\mathbf{m}^0 \in \Delta(N)$ and $\Delta(N)$ is convex. The convexity of a general set can only be established in an approximate-sense (cf. [7]). In contrast, we show that the convexity of a right-closed set can be determined in an exact sense. This leads to a computable test for the existence of a minimally restrictive invariant-based monitor that enforces liveness in this class of PNs.

The rest of the paper is organized as follows. We review the paradigm of supervisory- and monitor-based control for PNs in Section II after some notational preliminaries. Prior results that are relevant to this paper are reviewed in the final subsection of this section. The main results are presented in Section III, which is followed by an illustrative example in Section IV. Section V contains the conclusions.

II. Notations, Definitions and Some Preliminary Observations

$\mathcal{N}$ ($\mathcal{N}^+$) denotes the set of non-negative (positive) integers. The set of reals is denoted by $\mathcal{R}$. For $m, n, k \in \mathcal{N}^+$, the set of $m \times n$ matrices (k-dimensional vectors) of non-negative integers is represented as $\mathcal{N}^{m\times n}$ ($\mathcal{N}^k$). Similarly, $\mathcal{R}^{m\times n}$ and $\mathcal{R}^k$ denotes the set of $m\times n$ real-valued matrices and k-dimensional real-valued vectors, respectively. The term $card(\bullet)$ denotes the cardinality of the set argument.

Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{R}^n$, and $\lambda_1, \ldots, \lambda_k \in \mathcal{R}$. Then $\sum_{i=1}^k \lambda_i \mathbf{x}_i$ is a *convex combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{R}^n$ if $\forall i, \lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. The *Minkowski sum* of $\mathcal{A} \subseteq \mathcal{R}^n$ and $\mathcal{B} \subseteq \mathcal{R}^n$ is the set $\{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$.

For $i, j \in \mathcal{N}$, $\mathbf{A} \in \mathcal{R}^{i \times j}$, $\mathbf{b} \in \mathcal{R}^i$, a *polyhedron* $P(\mathbf{A}, \mathbf{b}) \subseteq \mathcal{R}^j$ is described as $P(\mathbf{A}, \mathbf{b}) := \{\mathbf{x} \in \mathcal{R}^j \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$. The polyhedron $P(\mathbf{A}, \mathbf{b})$ is a *rational polyhedron* if the entries of $\mathbf{A}$ and $\mathbf{b}$ are rational. The set of *integral points* (or, *lattice points*) in $P(\mathbf{A}, \mathbf{b})$ is denoted by $Int(P(\mathbf{A}, \mathbf{b}))(= P(\mathbf{A}, \mathbf{b}) \cap \mathcal{N}^j)$. If every component of every member in a polyhedron is non-negative, we say the polyhedron is in the *positive orthant*.

A *polytope* is a convex hull of a finite set of *vertices* in $\mathcal{R}^n$, which is a bounded polyhedron. The *Affine Minkowski-Weyl Duality theorem* states that a subset $P \subseteq \mathcal{R}^n$ is a polyhedron if and only if it is the Minkowski sum of a polytope and a finitely generated cone (cf. Section 5.3, [8]).

A set of vectors $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \geq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 \in \mathcal{M}))$. The set $\mathcal{M} \subseteq \mathcal{N}^n$ is identified by a finite set of minimal-elements, $\min(\mathcal{M}) \subset \mathcal{M}$. That is, $(\mathbf{m}^1 \in \min(\mathcal{M})) \wedge (\mathbf{m}^2 \in \mathcal{M}) \wedge (\mathbf{m}^2 \leq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 = \mathbf{m}^1)$. We say $\mathcal{M}$ is *convex* if and only if there is a convex set $\mathbf{C} \subseteq \mathcal{R}^n$ such that $\mathcal{M} = Int(\mathbf{C})$.

A *Petri net structure* $N = (\Pi, T, \Phi, \Gamma)$ is an ordered 4-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, \ldots, t_m\}$ is a collection of m *transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, and $\Gamma : \Phi \rightarrow \mathcal{N}^+$ is the *weight* associated with each arc. The *initial marking function* (or the *initial marking*) of a PN structure N is a function $\mathbf{m}^0 : \Pi \rightarrow \mathcal{N}$, which identifies the number of *tokens* in each place. We will use the term *Petri net* (PN) and the symbol $N(\mathbf{m}^0)$ to denote a PN structure N along with its initial marking $\mathbf{m}^0$. If $\forall \phi \in \Phi, \Gamma(\phi) = 1$, we say the PN has an *ordinary* structure. Otherwise, the PN is said to have a *general* structure.

For a string of transitions $\sigma \in T^*$, we use $\mathbf{x}(\sigma)$ to denote the *Parikh vector* of σ. That is, the i-th entry, $\mathbf{x}_i(\sigma)$, corresponds to the number of occurrences of transition t_i in σ.

A marking $\mathbf{m} : \Pi \rightarrow \mathcal{N}$ is sometimes represented by an integer-valued vector $\mathbf{m} \in \mathcal{N}^n$, where the i-th component $\mathbf{m}_i$ represents the token load ($\mathbf{m}(p_i)$) of the i-th place. The weight of an arc is represented by an integer that is placed along side the arc. For brevity, we refrain from denoting the weight of those arcs $\phi \in \Phi$ where $\Gamma(\phi) = 1$.

We define the set ${}^\bullet x := \{y \mid (y, x) \in \Phi\}$. A transition $t \in T$ is said to be *enabled* at a marking $\mathbf{m}^i$ if $\forall p \in {}^\bullet t, \mathbf{m}^i(p) \geq \Gamma((p, t))$. The set of enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$. An enabled transition $t \in T_e(N, \mathbf{m}^i)$ can *fire*, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$ according to $\mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \Gamma(p, t) + \Gamma(t, p)$.

In those contexts where the marking is interpreted as a non-negative integer-valued vector, it is useful to define the *input matrix* **IN** and *output matrix* **OUT** as two $n \times m$ matrices, whose (i, j)-th entry is defined as follows: $\mathbf{IN}_{i,j} = \Gamma((p_i, t_j))$ if $p_i \in {}^\bullet t_j$, and $\mathbf{IN}_{i,j}$ is zero otherwise; likewise, $\mathbf{OUT}_{i,j} = \Gamma((t_j, p_i))$ if $p_i \in t_j^\bullet$, and is zero otherwise. The *incidence matrix* $\mathbf{C}$ of the PN N is an $n \times m$ matrix, where $\mathbf{C} = \mathbf{OUT} - \mathbf{IN}$.

A. *Supervisory Control of PNs*

The set of transitions in the PN is partitioned into controllable- ($T_c \subseteq T$) and uncontrollable-transitions ($T_u \subseteq T$). The controllable (uncontrollable) transitions are represented as filled (unfilled) boxes in graphical representation of PNs.

A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \times T \rightarrow \{0, 1\}$, is a function that returns a 0 or 1 for each transition and each reachable marking. The supervisory policy $\mathcal{P}$ permits the firing of transition t_j at marking $\mathbf{m}^i$, if and only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $t_j \in T_e(N, \mathbf{m}^i)$ ($\mathcal{P}(\mathbf{m}^i, t_j) = 1$) for some marking $\mathbf{m}^i$, we say the transition t_j is *state-enabled* (*control-enabled*) at $\mathbf{m}^i$. A transition has to be state- and control-enabled before it can fire. To reflect the fact that the supervisory policy does not control-disable any uncontrollable transition, we assume that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$.

A string of transitions $\sigma = t_1 \cdots t_k$, where $t_j \in T (j \in \{1, \ldots, k\})$ is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$, if, (1) $t_1 \in T_e(N, \mathbf{m}^i), \mathcal{P}(\mathbf{m}^i, t_1) = 1$, and (2) for $j \in \{1, \ldots, k-1\}$ the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{j+1}) = 1$. If $\mathbf{m}^j$ results from the firing of $\sigma \in T^*$ starting from the initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^j$. The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$.

A transition t_k is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$. A policy $\mathcal{P}$ is a *liveness enforcing supervisory policy* (LESP) for $N(\mathbf{m}^0)$ if all transitions in $N(\mathbf{m}^0)$ are live under $\mathcal{P}$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if for every LESP $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \rightarrow \{0, 1\}$ for $N(\mathbf{m}^0)$, the following condition holds $\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t)$.

The set $\Delta(N) = \{\mathbf{m}^0 \mid \exists$ a LESP for $N(\mathbf{m}^0)\}$ represents the set of initial markings for which there is a LESP for a PN structure N. The set $\Delta(N)$ is *control invariant* (cf. [?], [9]) with respect to N. That is, if $\mathbf{m}^1 \in \Delta(N), t_u \in T_e(N, \mathbf{m}^1) \cap T_u$ and $\mathbf{m}^1 \xrightarrow{t_u} \mathbf{m}^2$ in N, then $\mathbf{m}^2 \in \Delta(N)$.

The set $\Delta(N)$ is not computable in general, but it can be computed using the software tool of reference [6], when N belongs to the class of (1) arbitrary PNs where $T_u = \emptyset$ [10], (2) ordinary FCPNs [3], or (3) the collection of general PN structures identified in reference [5], [4], [2].

Suppose $\mathbf{m}^0 \in \Delta(N)$, then the supervisory policy, $\mathcal{P}^*$, that control-disables any (controllable) transition at a marking in $\Delta(N)$ if and only if its firing would result in a new marking that is not in $\Delta(N)$, is the minimally restrictive LESP for $N(\mathbf{m}^0)$. There is *no* LESP, irrespective of the implementation-paradigm that is chosen, that is "better" than $\mathcal{P}^*$.

For any right-closed $\widetilde{\Delta}(N) \subseteq \Delta(N)$, the software described in reference [6] can be used to (a) test the control invariance of $\widetilde{\Delta}(N)$ with respect to a PN structure N, and (b) test if $\forall \widetilde{\mathbf{m}}_i \in \min(\widetilde{\Delta}(N))$, there is a closed-path in the *coverability-graph* (cf. pp. 438, [3]) of the PN $N(\widetilde{\mathbf{m}}_i)$, identified by $\sigma \in T^*$, such that $\mathbf{C}\mathbf{x}(\sigma) \geq \mathbf{0}$ (cf. the path-requirement in Lemma 5.15 of reference [3]). When these requirements are satisfied, the supervisory policy $\widetilde{\mathcal{P}}$, which disables a controllable transition at a marking if and only if its firing would result in a new marking that is not in $\widetilde{\Delta}(N)$, is a LESP [3]. This fact is implicitly used in the proof of Theorem III.3.

If $\Delta(N) - \widetilde{\Delta}(N) \neq \emptyset$, the minimally restrictive LESP $\mathcal{P}^*$ is applicable to a larger set of initial markings than the LESP $\widetilde{\mathcal{P}}$. Additionally, if $\exists \widehat{\mathbf{m}}^1 \in \widetilde{\Delta}(N), \exists \widehat{\mathbf{m}}^2 \in \Delta(N) - \widetilde{\Delta}(N), \exists t_c \in T_c$ such that $\widehat{\mathbf{m}}^1 \xrightarrow{t_c} \widehat{\mathbf{m}}^2$ in N, then unlike $\mathcal{P}^*$, the LESP $\widetilde{\mathcal{P}}$ will not permit the firing of t_c at $\widehat{\mathbf{m}}^1$. That is, the LESP $\widetilde{\mathcal{P}}$ will be more restrictive than the minimally restrictive LESP $\mathcal{P}^*$.

Liveness Enforcement using Monitors: Let $N(\mathbf{m}^0)$ be a PN, where $N = (\Pi, T, \Phi, \Gamma)$. The structure N can be augmented with the addition of extra places $\Pi_c = \{c_1, \ldots, c_k\}$ ($\Pi \cap \Pi_c = \emptyset$), or *monitors*, along with extra arcs $\Phi_c \subseteq (\Pi_c \times T) \cup (T \times \Pi_c)$ and their associated weights $\widehat{\Gamma} : \Phi_c \rightarrow \mathcal{N}^+$, to form a new structure $N_c = (\Pi \cup \Pi_c, T, \Phi \cup \Phi_c, \Gamma_c)$, where $\Gamma_c(\phi) = \Gamma(\phi)$ if $\phi \in \Phi$, and $\Gamma_c(\phi) = \widehat{\Gamma}(\phi))$ if $\phi \in \Phi_c$.

In subsequent text, when we deal with markings of N_c as $(n + k)$-dimensional vectors, we suppose the members of the place set of N_c are ordered as follows $\{p_1, \ldots, p_n, c_1, \ldots, c_k\}$, where $\Pi = \{p_1, \ldots, p_n\}$ and $\Pi_c = \{c_1, \ldots, c_k\}$. The initial token load of the monitors in Π_c are determined from the initial marking $\mathbf{m}^0$, according to $\Theta : \mathcal{N}^n \rightarrow \mathcal{N}^k$. The PN structure N_c with an initial marking of $((\mathbf{m}^0)^T \; \Theta(\mathbf{m}^0)^T)^T$ is represented as $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$. The set of markings that can be reached from the initial marking $((\mathbf{m}^0)^T \; \Theta(\mathbf{m}^0)^T)^T$ in N_c is denoted by $\Re(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$. Following the aforementioned convention, each $\mathbf{m} \in \Re(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$ can be interpreted as $\mathbf{m} = (\mathbf{m}_1^T \; \mathbf{m}_2^T)^T$, where the vector $\mathbf{m}_1 \in \mathcal{N}^n$ ($\mathbf{m}_2 \in \mathcal{N}^k$) corresponds to the token load of places in Π (Π_c). Since there might be arcs in Φ_c that originate from some $c_i \in \Pi_c$ to some uncontrollable transition $t_u \in T_u$, we must require $\forall \mathbf{m} \in \Re(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$, $(\forall p \in ({}^\bullet t_u \cap \Pi),\ \mathbf{m}(p) \geq \Gamma_c((p, t_u))) \Rightarrow (\forall c \in ({}^\bullet t_u \cap \Pi_c), \mathbf{m}(c) \geq \Gamma_c((c, t_u)))$. That is, no uncontrollable transition is prevented from firing at some marking that is reachable in $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$ due to a lack of sufficient tokens in the monitors. The requirement, $(\Pi_c \times T_u) \cap \Phi_c = \emptyset$, is sufficient but not necessary, for the above condition to be true (cf. figure 1(b)).

For $\mathbf{A} \in \mathcal{N}^{k \times n}$, $\mathbf{b} \in \mathcal{N}^k$, an initial marking $\mathbf{m}^0 \in \mathcal{N}^n$ where $\mathbf{A}\mathbf{m}^0 \geq \mathbf{b}$, and $\Theta(\mathbf{m}^0) = \mathbf{A}\mathbf{m}^0 - \mathbf{b}$, an *invariant-based* monitor ensures $\forall (\mathbf{m}_1^T \; \mathbf{m}_2^T)^T \in \Re(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0)), \mathbf{A}\mathbf{m}_1 \geq \mathbf{b}$ and $\mathbf{m}_2 = \mathbf{A}\mathbf{m}_1 - \mathbf{b} \geq \mathbf{0}$ [11]. That is, $\forall (\mathbf{m}_1^T \; \mathbf{m}_2^T)^T \in \Re(N_c, \mathbf{m}^0, \Theta(\mathbf{m}^0))$, the property $\mathbf{A}\mathbf{m}_1 \geq \mathbf{b}$, remains *invariant* for all reachable markings. When applicable, the invariant-based monitor is defined by the monitor incidence-matrix $\mathbf{AC}$, where $\mathbf{C}$ is the incidence matrix of the original PN structure N.

Liveness enforcement using invariant-based monitors seeks to augment the PN $N(\mathbf{m}^0)$ as described above, such that $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$ is live. When this objective is achieved, the influence of the monitors can be interpreted as an implicit definition of a LESP for the PN $N(\mathbf{m}^0)$.

This paper is about necessary and sufficient conditions under which there is an invariant-based monitor that ensures the liveness of $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$, which is equivalent to the minimally restrictive LESP $\mathcal{P}^*$ that ensures $\forall \mathbf{m}^0 \in \Delta(N)$, $\Re(N, \mathbf{m}^0, \mathcal{P}^*) \subseteq \Delta(N)$.

B. Review of Relevant Prior Work

We present a brief review of results that are pertinent to the approach used in this paper. Giua [12] introduced *monitors* into supervisory control of PNs. Moody and Antsaklis [11] used monitors to enforce liveness in certain classes of PNs, this work was extended by Iordache and Antsaklis [13]. Reveliotis and Nazeem [14] develop a necessary and sufficient condition for the existence of a single-tier set of linear inequalities defined on a appropriately defined vector-space, that identifies livelock-free states in a finite-state model of a livelock-prone RAS, with no uncontrollable events. This classifier, which is computed off-line, can be used to synthesize the minimally restrictive LESP for this class of fully-controlled RAS models. For those RAS that violate the conditions identified in reference [14], Cordone et. al [15] identify a sufficient condition for fully-controllable, finite-state RASs where the livelock-free states can be identified by a two-tiered, set of linear inequalities. Basile et al. [16] present sufficient conditions for minimally-restrictive, closed-loop liveness of a class of *Marked Graph* PNs supervised by monitors that enforce *Generalized Mutual Exclusion Constraints* (GMECs). Reference [17] presents a necessary and sufficient condition for the existence of GMECs that enforces, among other things, liveness, in a bounded PN.

Some of the synthesis procedures for invariant-based monitors for liveness enforcement outlined in above references do not consider the presence of uncontrollable transition in PN models. Other procedures require that there are no arcs from monitor places to uncontrollable transitions (cf. [11], [16], [13] for example), which is unnecessary (cf. the example in figure 1(c)).

If and when it exists, an invariant-based monitor for liveness enforcement is not necessarily minimally restrictive, in the general setting. That is, there could be a LESP, that permits the firing of a controllable transition at a marking, while the invariant-based monitor that enforces liveness unnecessarily prevents the controllable transition from firing at the same marking. The identification of a computable, necessary and sufficient condition for the existence of a minimally restrictive, invariant-based monitor that enforces liveness for the class of PNs with a right-closed $\Delta(N)$-set is the main contribution of this paper, which is presented in the next section.

III. Main Results

As noted in Section II-A, a right-closed set of markings $\widetilde{\Delta}(N) \subseteq \mathcal{N}^{card(\Pi)}$ that is control-invariant with respect to a PN structure $N = (\Pi, T, \Phi, \Gamma)$, that also satisfies the path-condition of Lemma 5.15 of reference [3], defines a LESP $\widetilde{\mathcal{P}}$. This observation is implicitly used in the proof of Theorem III.3, which notes that there is an invariant-based monitor that is equivalent to $\widetilde{\mathcal{P}}$ if and only if $\widetilde{\Delta}(N)$ is convex. Lemmas III.1 and III.2 find use in the proof of this result. Theorem III.6, which follows from Lemmas III.4 and III.5, implicitly identifies a convexity-test for an arbitrary right-closed set of markings. Proposition III.7 presents an explicit convexity-test for an arbitrary right-closed set.

The following Lemma is about a polyhedral representation of an arbitrary right-closed set of integral points. Since we concern ourselves with polyhedra that are in the positive orthant, we implicitly require $\forall \mathbf{x} \in P(\mathbf{A}, \mathbf{b}), \mathbf{x} \geq \mathbf{0}$ in this section.

Lemma III.1. *For $i, j \in \mathcal{N}$, $\mathbf{A} \in \mathcal{R}^{i \times j}$, $\mathbf{b} \in \mathcal{R}^i$, the set of integral points, $Int(P(\mathbf{A}, \mathbf{b})) = P(\mathbf{A}, \mathbf{b}) \cap \mathcal{N}^n$, in a polyhedron $P(\mathbf{A}, \mathbf{b})$ is right-closed if and only if $\mathbf{A}$ is non-negative.*

Proof. (*Only If*) Suppose $Int(P(\mathbf{A}, \mathbf{b}))$ is right-closed and $\mathbf{A}_{l,m}$, the (l, m)-th entry of $\mathbf{A}$, is negative. From the right-closure of $Int(P(\mathbf{A}, \mathbf{b}))$, we have $(\mathbf{x} \in Int(P(\mathbf{A}, \mathbf{b}))) \wedge (\widehat{\mathbf{x}} \geq \mathbf{x}) \Rightarrow (\widehat{\mathbf{x}} \in Int(P(\mathbf{A}, \mathbf{b})))$. Suppose, $\widehat{\mathbf{x}}_k = \mathbf{x}_k, \forall k \in \{1, 2, \ldots, j\} - \{m\}$, and if $\widehat{\mathbf{x}}_m$ is made arbitrarily large compared to $\mathbf{x}_m$, the l-th component of $\mathbf{A}\widehat{\mathbf{x}}$ can be made less than $\mathbf{b}_m$. This would mean $\widehat{\mathbf{x}} \notin Int(P(\mathbf{A}, \mathbf{b}))$. A contradiction.

(*If*) Follows directly from the definition of $Int(P(\mathbf{A}, \mathbf{b}))$. □

Additionally, without loss in generality $\mathbf{b}$ can be assumed to be non-negative if $P(\mathbf{A}, \mathbf{b})$ is in the positive orthant, as with a polyhedron whose integral points defines a set of markings of a PN. If $\mathbf{A}$ is a non-negative matrix and $\mathbf{b}_l < 0$. Then $\mathbf{b}_l \leftarrow 0$ will yield the same polyhedron as before. Consequently, $\mathbf{b}$ can be assumed to be non-negative as well for these instances.

Lemma III.2. *A right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$ is convex if and only if $\mathcal{M} = Int(P(\mathbf{A}, \mathbf{b}))$, for a polyhedron for some non-negative $\mathbf{A}$ and a non-negative $\mathbf{b}$.*

Proof. (If) If $\mathbf{A}$ and $\mathbf{b}$ are non-negative, then $Int(P(\mathbf{A}, \mathbf{b}))$ is a (convex) right-closed set (cf. Lemma III.1).

(Only If) A convex right-closed set $\mathcal{M}$ can be written as the set of integral points in a set $\mathcal{A}$ that is the Minkowski sum of the convex combination of the members of $\min(\mathcal{M})$ and the cone generated by the unit-vectors. From the *Affine Minkowski-Weyl Duality Theorem* we infer $\mathcal{A}$ is a polyhedron. From Lemma III.1 and the discussion that followed it, we can assume the polyhedron is of the form $P(\mathbf{A}, \mathbf{b})$, where $\mathbf{A}$ and $\mathbf{b}$ are non-negative. □

If $\mathcal{M} \subseteq \mathcal{N}^n$ is a right-closed set that is not convex, there can be no polyhedron $P(\mathbf{A}, \mathbf{b})$ such that $\mathcal{M} = Int(P(\mathbf{A}, \mathbf{b}))$. Or, any right-closed, polyhedral approximation to a non-convex, right-closed $\mathcal{M} \subseteq \mathcal{N}^n$ will inevitably exclude some members.

The following result is about a necessary and sufficient condition for the existence of an invariant-based monitor that is equivalent to the LESP $\widetilde{\mathcal{P}}$ that ensures the set of reachable markings under its supervision stays within some subset $\widetilde{\Delta}(N) \subseteq \Delta(N)$.

Theorem III.3. *Let $\widetilde{\Delta}(N) \subseteq \Delta(N)$ be a right-closed set that is control invariant with respect to a PN structure $N = (\Pi, T, \Phi, \Gamma)$. Further, let us suppose that each member of $\min(\widetilde{\Delta}(N))$ meets the path-requirement of Lemma 5.15 of reference [3]. As noted in section II-A, for any $\mathbf{m}^0 \in \widetilde{\Delta}(N)$, the supervisory policy $\widetilde{\mathcal{P}}$ that ensures $\Re(N, \mathbf{m}^0, \widetilde{\mathcal{P}}) \subseteq \widetilde{\Delta}(N)$, is an LESP. There exists an invariant-based monitor that is equivalent to the LESP $\widetilde{\mathcal{P}}$, if and only if $\widetilde{\Delta}(N)$ is convex.*

Proof. (If) If the right-closed set $\widetilde{\Delta}(N)$ is convex, then from Lemma III.2, it can be represented as, $Int(P(\mathbf{A}, \mathbf{b}))$, the set of integral points in a polyhedron $P(\mathbf{A}, \mathbf{b})$ in the positive orthant. From the results in references [12], [11], the invariant-based monitor $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$, where $\Theta(\mathbf{m}^0) = \mathbf{A}\mathbf{m}^0 - \mathbf{b}$, and an incidence matrix for the monitors that is given by $\mathbf{AC}$, will guarantee $\forall (\mathbf{m}_1^T \ \mathbf{m}_2^T)^T \in \Re(N, \mathbf{m}^0, \Theta(\mathbf{m}^0)), \mathbf{A}\mathbf{m}_1 \geq \mathbf{b}$, and $\mathbf{m}_2^T = \mathbf{A}\mathbf{m}_1 - \mathbf{b} \geq \mathbf{0}$.

The control-invariance of $\widetilde{\Delta}(N)$ with respect to N guarantees that any firing of a state-enabled uncontrollable transition at some marking in $\widetilde{\Delta}(N)$ in N will always result in a marking that is in $\widetilde{\Delta}(N)$. Consequently, in $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$, if there are any arcs from a monitor to an uncontrollable transition, it will not be prevented from firing due to insufficient tokens in the monitor (cf. Figure 1(b) for an illustrative example). If there is a controllable transition with a monitor as one of its input places, that is prevented from firing at some marking, then it must be that the new marking that would result if the controllable transition were permitted to fire would violate the (invariance) requirement $\mathbf{A}\mathbf{m}_1 \geq \mathbf{b}$, and $\mathbf{m}_1 \notin \widetilde{\Delta}(N)$. Therefore, the control exerted by the monitors in $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$ is the same as the LESP $\widetilde{\mathcal{P}}$.

(Only If) Suppose there is an invariant-based monitor $N_c(\mathbf{m}^0, \Theta(\mathbf{m}^0))$, with i-many monitors, that is equivalent to the LESP $\widetilde{\mathcal{P}}$ that ensures $\Re(N, \mathbf{m}^0, \widetilde{\mathcal{P}}) \subseteq \widetilde{\Delta}(N)$. Let us also suppose that the incidence matrix for the monitor places is given by $\mathbf{AC}$, and $\Theta(\mathbf{m}^0) = \mathbf{A}\mathbf{m}^0 - \mathbf{b}$ for any $\mathbf{m}^0$ such that $\mathbf{A}\mathbf{m}^0 \geq \mathbf{b}$, for appropriately defined $\mathbf{A} \in \mathcal{R}^{i \times n}$ and $\mathbf{b} \in \mathcal{R}^i$. Then, $\widetilde{\Delta}(N) = Int(P(\mathbf{A}, \mathbf{b}))$, which in turn implies $\widetilde{\Delta}(N)$ is convex. □

The following result follows directly from the definition of the convex-hull of a set of integral vectors.

Lemma III.4. *A set $\mathcal{M} \subseteq \mathcal{N}^n$ is convex if and only if $\mathcal{M} = Int(conv(\mathcal{M}))$.*

Proof. (Only if) Since $\mathcal{M}$ is convex, there exists a convex set $\mathbf{C} \subseteq \mathcal{R}^n$ such that $\mathcal{M} = Int(\mathbf{C})$. By definition, $conv(\mathcal{M})$ is the smallest convex set that contains all members of $\mathcal{M}$. Therefore, $conv(\mathcal{M}) \subseteq \mathbf{C}$, and $Int(conv(M)) \subseteq Int(\mathbf{C})(= \mathcal{M})$. Since $\mathcal{M} \subseteq Int(conv(\mathcal{M}))$, it follows that $\mathcal{M} = Int(conv(\mathcal{M}))$.

(If) Follows directly from the definition of convexity. □

Lemma III.5 proves that the set of integral vectors in the convex-hull of a right-closed set is also right-closed.

Lemma III.5. *The set $Int(conv(\mathcal{M}))$ is right-closed for any $\mathcal{M} \subseteq \mathcal{N}^n$ that is right-closed.*

Proof. Suppose $\widehat{\mathbf{m}} \in Int(conv(\mathcal{M}))$, then without loss of generality $\widehat{\mathbf{m}} = \sum_{i=1}^k \lambda_i \mathbf{m}_i$, where $\forall i \in \{1, 2, \ldots, k\}, \mathbf{m}_i \in \mathcal{M}$ and $\sum_{i=1}^k \lambda_i = 1$. If $\mathbf{1}_j \in \mathcal{N}^n$ is any one of the n-many unit vectors of $\mathcal{N}^n$, then $(\widehat{\mathbf{m}} + \mathbf{1}_j) = \sum_{i=1}^k \lambda_i (\mathbf{m}_i + \mathbf{1}_j)$, and $(\mathbf{m}_i + \mathbf{1}_j) \in \mathcal{M}$ as $\mathcal{M}$ is right-closed. Therefore, $(\widehat{\mathbf{m}} + \mathbf{1}_j) \in conv(\mathcal{M})$. The result follows from this observation. □

As a consequence of Lemma III.5, it follows that, $\min(Int(conv(\mathcal{M})))$, the set of minimal elements of $Int(conv(\mathcal{M}))$, is finite. Theorem III.6 yields an effectively computable test for the convexity of a right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$.

Theorem III.6. *A right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$ is convex if and only if $\min(\mathcal{M}) = \min(Int(conv(\mathcal{M})))$.*

Proof. If $\mathcal{M} \subseteq \mathcal{N}^n$ is right-closed, then $Int(conv(\mathcal{M}))$ is right-closed (cf. Lemma III.5), and is defined by $\min(Int(conv(\mathcal{M})))$. The result follows from Lemma III.4, and the fact that two right-closed sets are identical if and only if their minimal elements are identical. □

Consider a set $\mathcal{V} \subseteq \mathcal{N}^n$ that is the Minkowski sum of $\min(\mathcal{M})$ and the set $(\mathbf{0} \cup \{\mathbf{1}_i\}_{i=1}^n)$, where $\mathbf{1}_i$ is the i-th unit-vector, and $\mathbf{0}$ is the all-zero vector. That is, $\mathcal{V} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k\} \cup \left\{\bigcup_{p\in\{1,\ldots,k\},q\in\{1,\ldots,n\}}\{\mathbf{m}_p + \mathbf{1}_q\}\right\}$, where $\min(\mathcal{M}) = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k\}$. The following result shows that $\min(Int(conv(\mathcal{M}))) \subseteq conv(\mathcal{V})$. The finite number of integral vectors in the polytope $conv(\mathcal{V})$ can be enumerated using software packages like *Polymake*[1], which can be subsequently processed to compute the members of $\min(Int(conv(\mathcal{M})))$. Consequently, we have a procedure for testing the convexity of $\mathcal{M}$ (cf. theorem III.6).

Proposition III.7. *Let* $\min(\mathcal{M}) = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k\}$ *be the minimal elements of a right-closed set* $\mathcal{M} \subseteq \mathcal{N}^n$*. Then* $\min(Int(conv(\mathcal{M}))) \subseteq conv(\mathcal{V})$*, where* $\mathcal{V} = \min(\mathcal{M}) \cup \left\{\bigcup_{p\in\{1,\ldots,k\},q\in\{1,\ldots,n\}}\{\mathbf{m}_p + \mathbf{1}_q\}\right\}$.

Proof. Let $\mathbf{x} \in \min(Int(conv(\mathcal{M})))$, and $\mathbf{x} = (\sum_{i\in I}\lambda_i\mathbf{m}_i) + (\sum_{j\in J}\mu_j\mathbf{p}_j)$ for appropriate index sets I and J, where $\{\mathbf{p}_j\}_{j\in J}$ is a set of integral vectors in the set $(\mathcal{M} - \min(\mathcal{M}))$, $(\sum_{i\in I}\lambda_i + \sum_{j\in J}\mu_j) = 1$, $\forall i \in I, \lambda_i \geq 0$, and $\forall j \in J, \mu_j \geq 0$. Let us suppose each $\mathbf{p}_j > \widehat{\mathbf{m}}_j$ for $j \in J$, and $\widehat{\mathbf{m}}_j \in \min(\mathcal{M})$. Then $\mathbf{x} = (\sum_{i\in I}\lambda_i\mathbf{m}_i) + (\sum_{j\in J}\mu_j\mathbf{p}_j) \geq (\sum_{i\in I}\lambda_i\mathbf{m}_i) + (\sum_{j\in J}\mu_j\widehat{\mathbf{m}}_j) = \sum_{i\in K}\lambda_i\mathbf{m}_i = \mathbf{y}$, where the index set K is obtained from the index sets I and J through the appropriate operations.

Since $\mathbf{y} \leq \mathbf{x}$, and $\mathbf{x} \in \min(Int(conv(\mathcal{M})))$, we have $\mathbf{x} = \lceil\mathbf{y}\rceil$. For any pair of sets $\mathcal{A}$ and $\mathcal{B}$, $conv(\mathcal{A} + \mathcal{B}) = conv(\mathcal{A}) + conv(\mathcal{B})$, where the summation operator is the Minkowski sum. This, together with the fact that $\lceil\mathbf{y}\rceil = \mathbf{y} + \mathbf{w}$, where $\mathbf{w} \in conv(\{\mathbf{0}\} \cup \{\mathbf{1}_q\}_{q=1}^n)$ and $\mathbf{y} \in conv(\min(\mathcal{M}))$, it follows that $\mathbf{x} \in conv(\mathcal{V})$. □

We present two illustrative examples in the next section.

IV. Example

For the ordinary FCPN $N_1(\mathbf{m}_1^0)$ shown in figure 1(a), using the software of reference [6], we have $\min(\Delta(N_1)) = \{(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 0\ 0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 0\ 0\ 0\ 1\ 0)^T, (0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)^T\}$. Equivalently, $\Delta(N_1) = \{\mathbf{m} \in \mathcal{N}^8 \mid (1\ 0\ 0\ 1\ 0\ 1\ 1\ 1)\mathbf{m} \geq 1\}$. Since $\Delta(N_1)$ is convex, from Theorem III.3 we know there is a minimally restrictive, invariant-based monitor that enforces liveness in $N_1(\mathbf{m}_1^0)$ for any $\mathbf{m}_1^0 \in \Delta(N_1)$, which is shown in figure 1(a). This invariant-based monitor is equivalent to the minimally restrictive LESP $\mathcal{P}^*$ of section II-A that is synthesized for $N_1(\mathbf{m}_1^0)$.

Consider the plant PN $N_2(\mathbf{m}_2^0)$ shown in Figures 1(b) and 1(c), which is taken from Figure 2c of reference [13]. The set $\Delta(N_2)$ is right-closed (cf. the class of general PNs $\mathcal{F}$, [4]) and is characterized by the minimal elements: $\{(0\ 0\ 1\ 0)^T, (2\ 0\ 0\ 0)^T, (1\ 0\ 0\ 1)^T, (0\ 2\ 0\ 0)^T, (0\ 0\ 0\ 2)^T\}$. The minimally restrictive LESP for $N_2(\mathbf{m}_2^0)$ for any $\mathbf{m}_2^0 \in \Delta(N_2)$ is shown in Figure 1(b). We note that $\Delta(N)$ is not convex, as $\underbrace{(1\ 1\ 0\ 0)^T}_{\notin\Delta(N)} = \frac{1}{2}\times\underbrace{(2\ 0\ 0\ 0)^T}_{\in\Delta(N)} + \frac{1}{2}\times\underbrace{(0\ 2\ 0\ 0)^T}_{\in\Delta(N)}$, consequently from Theorem III.3, there can be no minimally restrictive, invariant-based monitor that enforces liveness for this PN. Additionally, any invariant-based monitor that enforces liveness in $N_2(\mathbf{m}_2^0)$ will either restrict itself to a set of initial markings that is a strict subset of $\Delta(N)$, and/or the control enforced by the monitors will be more restrictive than the LESP of Figure 1(b). To illustrate this, consider the right-closed set of markings, $\widehat{\Delta}(N_2) \subset \Delta(N_2)$, defined by the inequalities

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}_{\mathbf{A}}\mathbf{m} \geq \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\mathbf{b}}, \mathbf{m} \in \mathcal{N}^4,$$

which has thirteen minimal elements: $\{(1\ 0\ 1\ 0)^T$, $(0\ 0\ 2\ 0)^T$, $(3\ 0\ 0\ 0)^T$, $(0\ 1\ 0\ 2)^T$, $(0\ 1\ 1\ 0)^T$, $(1\ 0\ 0\ 2)^T$, $(2\ 0\ 0\ 1)^T$, $(0\ 2\ 0\ 1)^T$, $(1\ 1\ 0\ 1)^T$, $(2\ 1\ 0\ 0)^T$, $(1\ 2\ 0\ 0)^T$, $(0\ 3\ 0\ 0)^T$, $(0\ 0\ 1\ 1)^T\}$, and is control invariant with respect to N_2. With the help of the software described in reference [6], it can be verified that each of these thirteen minimal elements also meet the path-requirement of Lemma 5.15 of reference [3]. Therefore, for any $\mathbf{m}_2^0 \in \widehat{\Delta}(N_2)$, the supervisory policy that ensures the reachable markings stay within $\widehat{\Delta}(N_2)$ enforces liveness in $N(\mathbf{m}_2^0)$. Since $\widehat{\Delta}(N)$ is convex, following Theorem III.3, this supervisory policy can be equivalently represented by an invariant-based monitor $\widehat{N}_{2,c}(\mathbf{m}^0, \widehat{\Theta}(\mathbf{m}_2^0))$ shown in Figure 1(c).

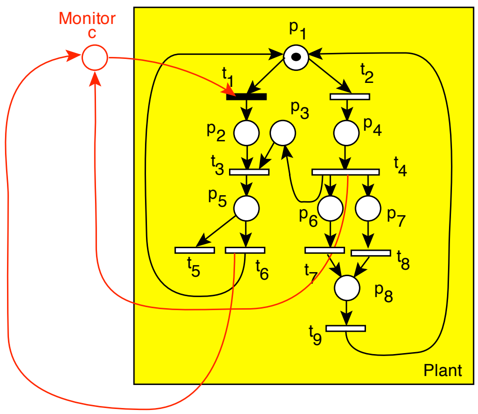
We call attention to the fact that even though there is an input arc from c_2 to the uncontrollable transition t_3 in $\widehat{N}_{2,c}(\mathbf{m}_2^0, \widehat{\Theta}(\mathbf{m}_2^0))$, it does not disable its firing at any marking that places at least two tokens in p_1. Additionally, the invariant-based monitor $\widehat{N}_{2,c}(\mathbf{m}_2^0, \widehat{\Theta}(\mathbf{m}_2^0))$ derived from $\widehat{\Delta}(N_2)$, is more restrictive than minimally restrictive LESP shown in Figure 1(b) derived from $\Delta(N_2)$.

V. Conclusions

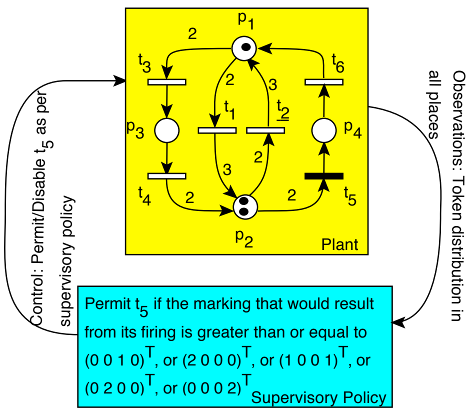
We considered a family of PN structures where for any instance N, the existence of a *liveness enforcing supervisory policy* (LESP) for $N(\mathbf{m}^0)$ is sufficient to infer there is a LESP for $N(\widehat{\mathbf{m}}^0)$, where $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$. Consequently, the set of initial markings for which there is a LESP, $\Delta(N)$, is *right-closed*. For $\mathbf{m}^0 \in \Delta(N)$, the minimally restrictive LESP for $N(\mathbf{m}^0)$ disables the firing of any controllable transition at a reachable marking, if and only if its firing would result in a new marking that is not in $\Delta(N)$.

Alternately, we can augment the structure of N with a few additional places, or *monitors*, along with extra arcs between the monitors and the transitions in N. The initial-marking of the monitors is uniquely determined by the initial marking $\mathbf{m}^0$ of N. An *invariant-based* monitor ensures the markings of the original PN stay within an appropriately defined polyhedron. While there might be an invariant-based monitor that enforces liveness in a PN, it is not guaranteed to be minimally restrictive, in general. That is, it might unnecessarily prevent the firing of some controllable transition at some reachable marking. We derived a computable, necessary and sufficient

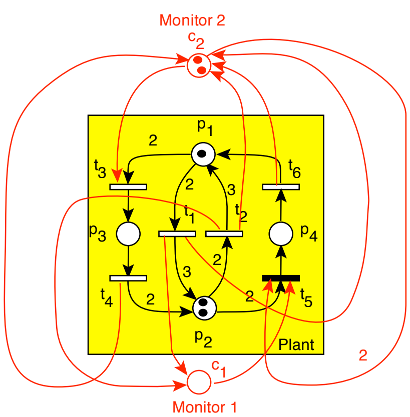
[1] http://polymake.org/doku.php.



(a) An invariant-based monitor $N_{1,c}(\mathbf{m}_1^0, \Theta_1(\mathbf{m}_1^0))$



(b) Right-Closed Set Based Supervision



(c) $\widehat{N}_{2,c}(\mathbf{m}^0, \widehat{\Theta}_2(\mathbf{m}_2^0))$

Fig. 1. (a) An invariant-based monitor $N_{1,c}(\mathbf{m}_1^0, \Theta_1(\mathbf{m}_1^0))$ where $\mathbf{m}_1^0 \in \Delta(N_1)(= \{\mathbf{m} \in \mathcal{N}^8 \mid (1\ 0\ 0\ 1\ 0\ 1\ 1\ 1)\mathbf{m} \geq 1\})$, and $\Theta_1(\mathbf{m}_1^0) = (1\ 0\ 0\ 1\ 0\ 1\ 1\ 1)\mathbf{m}_1^0 - 1$, that is live and minimally restrictive as well. (b) A minimally restrictive LESP for the general plant PN $N_2(\mathbf{m}_2^0)$. This LESP ensures the markings reachable under its supervision from any $\mathbf{m}_2^0 \in \Delta(N_2)$ remain within the right-closed set $\Delta(N_2)$, where $\min(\Delta(N_2)) = \{(0\ 0\ 1\ 0)^T, (2\ 0\ 0\ 0)^T, (1\ 0\ 0\ 1)^T, (0\ 2\ 0\ 0)^T, (0\ 0\ 0\ 2)^T\}$. There is no invariant-based monitor that is equivalent to this minimally restrictive LESP as $\Delta(N_2)$ is not convex (cf. theorem III.3). (c) An invariant-based monitor $\widehat{N}_{2,c}(\mathbf{m}_2^0, \widehat{\Theta}_2(\mathbf{m}_2^0))$ that enforces liveness for any $\mathbf{m}_2^0 \in \widehat{\Delta}(N_2)$, where $\widehat{\Delta}(N_2) = \{\mathbf{m} \in \mathcal{N}^4 \mid (1\ 1\ 2\ 1)\mathbf{m} \geq 3 \text{ and } (1\ 1\ 1\ 0)\mathbf{m} \geq 1\} (\subset \Delta(N_2))$. The function $\widehat{\Theta}_2(\mathbf{m}_2^0)$ assigns an initial token load of $\mathbf{m}_2^0(p_1) + \mathbf{m}_2^0(p_2) + 2\mathbf{m}_2^0(p_3) + \mathbf{m}_2^0(p_4) - 3$ and $\mathbf{m}_2^0(p_1) + \mathbf{m}_2^0(p_2) + \mathbf{m}_2^0(p_3) - 1$ to c_1 and c_2, respectively. Additionally, even though there is an arc from c_2 to the uncontrollable transition t_3, its firing is never inhibited when there are two or more token in p_1. Transition t_5 is not permitted to fire at the marking shown in this figure, while the same transition is permitted by the minimally restrictive LESP of Figure 1(b). Any synthesis procedure for an invariant-based monitor that enforces liveness that requires there be no arcs from a monitor to an uncontrollable transition as a constraint, will miss this construction.

condition for the existence of a minimally restrictive invariant-based monitor that enforces liveness in a PN N with a right-closed set $\Delta(N)$. In those cases where there is no minimally restrictive invariant-based monitor, this result can be used to synthesize a more restrictive invariant-based monitor that implicitly defines a LESP for $N(\mathbf{m}^0)$.

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