On Liveness Enforcing Supervisory Policies for Arbitrary Petri Nets

C. Chen[*‡] , A. Raman[† ‡] , H. Hu[*], R.S. Sreenivas[†]
[*] School of Electro-Mechanical Engineering, Xidian University
chenchen0@xidian.edu.cn, huhesuan@gmail.com
[†] Coordinated Science Laboratory & Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign
{raman12, rsree}@illinois.edu

***Abstract*—Neither the existence, nor the non-existence, of a *Liveness Enforcing Supervisory Policy* (LESP) for an arbitrary *Petri net* (PN) is semi-decidable. In an attempt to identify decidable instances, we explore the decidability of certain properties of the set of initial markings for which an LESP exists, and the decidability of the existence of a specific class of LESPs. We first prove that for an arbitrary PN structure, determining if there is an initial marking, or there are no initial markings, for which there is an LESP, is not semi-decidable. Then, we characterize the class of PN structures for which the set of all initial-markings for which an LESP exists is *right-closed*. We show that testing membership, or non-membership, of an arbitrary PN in this class of PNs is not semi-decidable. We then consider a restricted class of LESPs, called *marking monotone LESPs* (MM-LESPs). We show that the existence of an MM-LESP for an arbitrary PN is decidable.**

***Index Terms*—Petri Nets, Supervisory Control, Discrete Event Dynamic Systems.**

I. Introduction

A *Discrete Event Dynamic System* (DEDS) is a discrete state, event driven system, where the discrete-state changes at a discrete-time instant due to the occurrence of *events*. Manufacturing- and Service-Systems; Database-Systems; Traffic-Networks; Integrated Command, Control, Communication and Information (C^3I) systems; etc., are examples of DEDS. *Petri nets* (PNs) [1] are a popular modeling formalism for DEDS since they can provide abundant structural information about the system, and they are amenable to mathematical analysis.

A PN model is a directed bipartite graph where the two sets of nodes are referred to as *places* and *transitions*. The edges connecting the places with the transitions and vice-versa are referred to as *arcs*. The arcs have weights associated with them. The initial marking, $\mathbf{m}^0$, of the PN associates a non-negative, integer-valued token-load to each place. A PN $N(\mathbf{m}^0)$ is essentially the *PN-structure* N along with an *initial-marking* $\mathbf{m}^0$. A transition is said to be *state-enabled* if the token-load of each of its input places is no less than the weight associated with the arc from the place to the transition. A state-enabled transition could *fire*, which reduces (resp. increases) the token-load of each of its input (resp. output) places according to the associated arc weights. This process repeats at the newly created token-load distribution (marking), as often as necessary.

A PN is said to be *live* if it is possible to fire any transition, although not necessarily immediately, from any marking that is reachable from the *initial marking*. If a PN model of a DEDS is not live, it is of interest to investigate the existence of a *supervisory policy* that can make the supervised-PN live. The supervisory policy enforces liveness by preventing the firing of a subset of *controllable* transitions, which correspond to controllable activities (or events) of the DEDS. On the other hand, the *uncontrollable* transitions represent activities (or events) that are external to the DEDS, which cannot be prevented from occurring by the supervisory policy.

A *decision-problem*, that is posed as a "*yes*" or "*no*" question for each input, is *decidable* (resp. *undecidable*) if there exists (resp. does not exist) a single algorithm that correctly answers "*yes*" or "*no*" to all possible inputs. It is *semi-decidable* if there exists a single algorithm that will always correctly answer "*yes*", but does not return anything when the answer is "*no*". Every decision-problem has an associated *complementary* decision-problem. The answer to the complementary problem is "*yes*" if and only if the answer to the original decision problem is "*no*". A decision-problem is decidable if and only if the decision-problem *and* its complement, are semi-decidable (cf. section 1.2.2, [2]).

In this paper, we explore questions regarding what can, and cannot be done in the context of synthesizing LESPs for arbitrary PNs from a computability viewpoint. Specifically, for a PN structure N with n places, we are interested in understanding the nature of the set $\Delta(N)$ defined as follows:

$$\Delta(N) = \{\mathbf{m}^0 \in \mathcal{N}^n : \text{there exists an LESP for } N(\mathbf{m}^0)\}, \quad (1)$$

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[‡] Equal contribution by student authors. The student and faculty authors' names are arranged in alphabetical order of their last names respectively.

where $\mathcal{N}$ denotes the set of non-negative integers. The test for existence (resp. non-existence) of an LESP for an initial marking reduces to the decision-problem – "*Is* $\mathbf{m}^0 \in \Delta(N)$?" (resp. "*Is* $\mathbf{m}^0 \notin \Delta(N)$?"). Reference [3] proved that "*Is* $\mathbf{m}^0 \in \Delta(N)$?" is undecidable for arbitrary PNs by reducing it to the *Reachability Inclusion Problem* [4]. This result was further refined in [5]. Although undecidable for arbitrary PNs, there are classes of PNs, with certain structural properties, for which the existence of an LESP is decidable [5]–[8]. The $\mathcal{H}$-class of PN structures is the largest among the decidable classes identified in those references [7]. The $\mathcal{H}$-class has the following structural properties: (1) for each place, the weights associated with the outgoing arcs that terminate on uncontrollable transitions must be the smallest of all outgoing arc-weights; (2) the set of input places to each uncontrollable transition is no larger than the set of input places of any transition which shares a common input place with it. For these classes of PNs, $\Delta(N)$ is *right-closed*. That is, if there exists an LESP for an initial marking, then there exists a (possibly different) LESP for all term-wise larger initial markings as well.

If a transition is permitted to fire by a *marking monotone* policy (MM-policy) at a marking $\mathbf{m} \in \mathcal{N}^n$, then it will be permitted to fire at any marking $\widehat{\mathbf{m}} \geq \mathbf{m}$, as well. If an MM-policy that is an LESP for $N(\mathbf{m}^0)$ is also an LESP for $N(\widehat{\mathbf{m}}^0)$ for any $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$, then we say there is a *marking monotone LESP* (MM-LESP) for $N(\mathbf{m}^0)$. That is, if there is an MM-LESP for $N(\mathbf{m}^0)$, then there is an MM-LESP for $N(\widehat{\mathbf{m}}^0)$ for any $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$, which means the set

$$\Delta_M(N) = \{\mathbf{m}^0 \in \mathcal{N}^n : \text{ there exists an MM-LESP for } N(\mathbf{m}^0)\},$$

is right-closed, and $\Delta_M(N) \subseteq \Delta(N)$. Note that if $\Delta(N)$ is right-closed, then $\Delta_M(N) = \Delta(N)$ (because the firing of a transition results in a larger marking if it is fired from a larger initial marking). Coincidentally, if $N \in \mathcal{H}$, then $\Delta(N) = \Delta_M(N)$ [7]. These results in the literature provide pointers on a possible approach to expand the class of PNs for which the existence of an LESP is decidable. First, by restricting the properties of the set $\Delta(N)$ (for example, right-closure); and second, by restricting the nature of the LESP (for example, MM-LESPs).

With an objective of characterizing the structure and properties of PNs for which the existence and non-existence of an LESP is decidable, we start with investigating if it is the right-closure of $\Delta(N)$ that is the reason for the decidability of LESP. As the first contribution of the paper, we characterize the exhaustive class of PNs, $\widehat{\mathcal{H}}$, such that $(N \in \widehat{H}) \Leftrightarrow (\Delta(N)$ is right-closed). Testing membership in $\widehat{\mathcal{H}}$-class is posed as the decision problems: "*Is* $\Delta(N)$ *right-closed?*" and "*Is* $\Delta(N)$ *not right-closed?*". We then observe that an empty set is right-closed by definition. Consequently, a positive result for the decision problem "*Is* $\Delta(N)$ *right-closed?*" would mean that there are either countably-infinite markings or no markings for which an LESP exists. Therefore, before venturing into the decision problem of right-closure, we investigate the decision-problems: "*Is* $\Delta(N) = \emptyset$*?*" and "*Is* $\Delta(N) \neq \emptyset$*?*". In addition to being associated with right-closure, these can also be interpreted as a generalization of the decision problems: "*Is* $\mathbf{m}^0 \in \Delta(N)$*?*" and "*Is* $\mathbf{m}^0 \notin \Delta(N)$*?*" studied in [3] and [5]. As the second contribution of the paper, we show that "*Is* $\Delta(N) = \emptyset$*?*" and "*Is* $\Delta(N) \neq \emptyset$*?*" are not semi-decidable for arbitrary PNs.

Coming back to right-closure, as the third contribution, we prove that "*Is* $\Delta(N)$ *right-closed?*" is not decidable. Following this result, we introduce an extension to $\mathcal{H}$-class of PNs, the $\mathcal{K}$-class. The decision problems: "$N \in \mathcal{K}$?" and "$N \notin \mathcal{K}$?" are decidable. For $N \in \mathcal{K}$, $\Delta(N)$ is right-closed, and $\mathcal{K}$ is the largest characterized class of PNs for which $\Delta(N)$ is right-closed. We have the following inclusion relation between the various PN structures with a right-closed $\Delta(N)$: $\mathcal{H} \subset \mathcal{K} \subset \widehat{\mathcal{H}}$.

As the fourth contribution, we further reduce the scope of the problem and investigate a variation to right-closure. We attempt to determine that for a given PN N, if there exists a *subset* of markings, $\widetilde{\Delta}(N) \subseteq \Delta(N)$, that is right-closed. This relaxation does not improve the results, and the decision problems: "*Is there a right-closed subset of* $\Delta(N)$*?*" and "*Is there no right-closed subset of* $\Delta(N)$*?*" are also not semi-decidable.

As the last result in the paper, we turn our attention at restricting the nature of LESPs. We pose the decision problems: *"Is* $\mathbf{m}^0 \in \Delta_M(N)$?″ and *"Is* $\mathbf{m}^0 \notin \Delta_M(N)$?″ and prove that it is decidable. That is, the existence and non-existence of an MM-LESP for an arbitrary PN is decidable. Moreover, the algorithm for decidability also evaluates the largest $\Delta_M(N)$, if $\Delta_M(N) \neq \emptyset$.

Thus, starting from the two decision problems: *"Is* $\mathbf{m}^0 \in \Delta(N)$?*"* and *"Is* $\mathbf{m}^0 \notin \Delta(N)$?*"* that are not semi-decidable, we present a string of results that culminate in decidable sub-problems: *"Is* $\mathbf{m}^0 \in \Delta_M(N)$?*"* and *"Is* $\mathbf{m}^0 \notin \Delta_M(N)$?*"*. These results lead to the conclusion that extracting any kind of information about $\Delta(N)$ for an arbitrary PN is most likely an extremely hard problem. Besides, we can also conclude that between the properties of the set of initial markings for which an LESP exists, and the characteristics of the LESP, it is the characteristics of the LESP that plays a prominent role in determining decidability. To be specific, let $\Re(N, \mathbf{m}, \mathcal{P})$ denote the set of reachable markings for $N(\mathbf{m})$ under the supervision of an LESP $\mathcal{P}$. If a supervisory policy $\mathcal{P}$ is such that $\Re(N, \mathbf{m}, \mathcal{P})$ (which can have an unbounded number of markings) can be reduced to a reachability graph with a finite number of appropriately defined symbolic markings such that the liveness property is preserved, then the existence of $\mathcal{P}$ is likely to be decidable. We expand on this point in Section IX.

The paper is organized as follows: Section II presents the notations and definitions used in the paper. We present a necessary and sufficient condition for right-closure of $\Delta(N)$ for an arbitrary PN N in Section III. In Section IV, we prove that "*Is* $\Delta(N) = \emptyset$?" is not decidable. Using this result, we prove that "*Is* $\Delta(N)$ *right-closed*?" is not-decidable for an arbitrary PN N in Section V. Following this, in Section VI, we show that a variation of the earlier decision problem: "*Is there a right-closed subset of* $\Delta(N)$*?*" is not decidable. After introducing $\mathcal{K}$-class of PN structures in Section VII, in Section VIII we prove that the existence of a marking-monotone LESP for an arbitrary PN is decidable. We conclude the paper with Section IX.

II. Notations and Definitions

We use $\mathcal{N}$ ($\mathcal{N}^+$) to denote the set of non-negative (positive) integers. The term $card(\bullet)$ denotes the cardinality of the set argument. The symbol Σ^* denotes the set of all possible strings (including the empty string) that can be constructed from an alphabet Σ.

The unit vector whose i-th value is unity is represented as $\mathbf{1}_i$. Given two integer-valued vectors $\mathbf{x}, \mathbf{y} \in \mathcal{N}^k$, we use the notation $\mathbf{x} \geq \mathbf{y}$ if $\mathbf{x}_i \geq \mathbf{y}_i$ for all $i \in \{1, 2, \ldots k\}$. We use the term $\max\{\mathbf{x}, \mathbf{y}\}$ to denote the vector whose i-th entry is $\max\{\mathbf{x}_i, \mathbf{y}_i\}$. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{R}^n$, and $\lambda_1, \ldots, \lambda_k \in \mathcal{R}$, where $\mathcal{R}$ denotes the set of real numbers. Then $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ is a *convex combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{R}^n$ if $\forall i, \lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. The *Minkowski sum* of $\mathcal{A} \subseteq \mathcal{R}^n$ and $\mathcal{B} \subseteq \mathcal{R}^n$ is the set $\{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$. The *convex-hull* $conv(\{\mathbf{x}_1, \ldots, \mathbf{x}_k\})$ of a set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is the smallest convex set that contains it. We use the term $Int(\bullet)$ to denote the set of integer-valued vectors contained in the set argument. For instance, $Int(conv(\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}))$ denotes the set of integer-valued vectors in the convex hull of $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

A *Petri net structure* $N = (\Pi, T, \Phi, \Gamma)$ is an ordered 4-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, \ldots, t_m\}$ is a collection of m *transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, and $\Gamma : \Phi \to \mathcal{N}^+$ is the *weight* associated with each arc. The *initial marking function* (or the *initial marking*) of a PN structure N is a function $\mathbf{m}^0 : \Pi \to \mathcal{N}^n$, which identifies the number of *tokens* in each place. The marking can be interpreted as an integer-valued vector where the i-th component represents the token load of the i-th place $p_i \in \Pi$. For ease of exposition, some of the symbols that we have used to denote a marking are specific to that particular section. The meaning of a particular symbol should be clear from the context.

We use the notation $\mathbf{m}(p)$ to denote the tokens in place $p \in \Pi$. Let $\Pi_1 \subseteq \Pi_2 \subseteq \Pi$, $\mathbf{m}^1 \in \mathcal{N}^{card(\Pi_1)}$ and $\mathbf{m} \in \mathcal{N}^{card(\Pi_2)}$. We use the notation $\mathbf{m}(\Pi_1) = \mathbf{m}^1$ to denote $\mathbf{m}(p) = \mathbf{m}^1(p)$, for all $p \in \Pi_1$.

We will use the term *Petri net* (PN) and the symbol $N(\mathbf{m}^0)$ to denote a PN structure N along with its initial marking $\mathbf{m}^0$. In graphical representations of PNs, the places are represented by circles, transitions by rectangles, and arcs are represented by directed edges. For brevity, only the non-unitary arc-weights are placed alongside arcs in graphic representations of PNs in this paper. The tokens are represented by filled-circles that reside in the circles that represent places. The set of transitions in the PN is partitioned into controllable- ($T_c \subseteq T$) and uncontrollable-transitions ($T_u \subseteq T$). The controllable (uncontrollable) transitions are represented as filled (unfilled) boxes in graphical representation of PNs.

We define the sets ${}^\bullet x = \{y | (y, x) \in \Phi\}$ and $x^\bullet = \{y | (x, y) \in \Phi\}$. A transition $t \in T$ is said to be *state-enabled* at a marking $\mathbf{m}^i$ if $\forall p \in {}^\bullet t, \mathbf{m}^i(p) \geq \Gamma(p, t)$. The set of state-enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$.

If $t_j \in T_e(N, \mathbf{m})$, then $\mathbf{m} \geq \mathbf{IN}_{\bullet, j}$, which is the j-th column of the $n \times m$ *input matrix* $\mathbf{IN}$, defined as

$$\mathbf{IN}_{i,j} = \begin{cases} \Gamma(p, t) & \text{if } p_i \in {}^\bullet t_j, \\ 0 & \text{otherwise.} \end{cases}$$

The *output matrix* is an $n \times m$ matrix that encodes the firing of an enabled transition:

$$\mathbf{OUT}_{i,j} = \begin{cases} \Gamma(t, p) & \text{if } p_i \in t_j^\bullet, \\ 0 & \text{otherwise.} \end{cases}$$

The *incidence matrix* $\mathbf{C}$ of the PN N is an $n \times m$ matrix, where $\mathbf{C} = \mathbf{OUT} - \mathbf{IN}$.

A supervisory policy $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$, is a function that returns a 0 or 1 for each marking and each transition. The supervisory policy $\mathcal{P}$ permits the firing of transition t_j at marking $\mathbf{m}^i$, if and only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $\mathcal{P}(\mathbf{m}^i, t_j) = 1$ for some marking $\mathbf{m}^i$, we say the transition t_j is *control-enabled* at $\mathbf{m}^i$. A transition has to be state- and control-enabled before it can fire. To reflect the fact that the supervisory policy does not control-disable any uncontrollable transition, we assume that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$. A state- and control-enabled transition t can fire, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$ according to $\mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \Gamma(p, t) + \Gamma(t, p)$.

A string of transitions $\sigma = t_1 \ldots t_k$, where $t_j \in T$ ($j \in \{1, \ldots, k\}$), is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$ if 1) the transitions $t_1 \in T_e(N, \mathbf{m}^i)$, $\mathcal{P}(\mathbf{m}^i, t_1) = 1$, and 2) for $j \in \{1, 2, \ldots, k-1\}$, the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{j+1}) = 1$. If $\mathbf{m}^{i+k}$ results from the firing of $\sigma \in T^*$ starting from the initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^{i+k}$. If $\mathbf{x}(\sigma)$ is an m-dimensional vector whose i-th component corresponds to the number of occurrences of t_i in a valid string σ, and if $\mathbf{m}^i \xrightarrow{\sigma} \mathbf{m}^j$, then $\mathbf{m}^j = \mathbf{m}^i + \mathbf{C}\mathbf{x}(\sigma)$.

Given an initial marking $\mathbf{m}^0$, the set of *reachable markings* for $\mathbf{m}^0$, which is denoted by $\Re(N, \mathbf{m}^0)$, is defined as the set of markings generated by all valid firing strings starting with marking $\mathbf{m}^0$ in the PN N. The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$.

A PN $N(\mathbf{m}^0)$ is said to be *live* if $\forall t \in T, \forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i)$ such that $t \in T_e(N, \mathbf{m}^j)$ (cf. *level 4 liveness*, [1], [9]). A transition t_k is *live* under the supervision of $\mathcal{P}$, if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$. A policy $\mathcal{P}$ is a *liveness enforcing supervisory policy* (LESP) for $N(\mathbf{m}^0)$ if all transitions in $N(\mathbf{m}^0)$ are live under $\mathcal{P}$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if for every LESP $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \to \{0, 1\}$ for $N(\mathbf{m}^0)$, the following condition holds: $\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t)$. The set

$$\Delta(N) = \{\mathbf{m}^0 : \exists \text{ an LESP for } N(\mathbf{m}^0)\}$$

represents the set of initial markings for which there is an LESP for a PN structure N. The set $\Delta(N)$ is *control invariant* with respect to N. That is, if $\mathbf{m}^1 \in \Delta(N), t_u \in T_e(N, \mathbf{m}^1) \cap T_u$ and $\mathbf{m}^1 \xrightarrow{t_u} \mathbf{m}^2$ in N, then $\mathbf{m}^2 \in \Delta(N)$. Equivalently, only the firing of a controllable transition at any marking in $\Delta(N)$ can result in a new marking that is not in $\Delta(N)$. There is an LESP for $N(\mathbf{m}^0)$ if and only if $\mathbf{m}^0 \in \Delta(N)$. If $\mathbf{m}^0 \in \Delta(N)$, the LESP that prevents the firing of a controllable transition at any marking when its firing would result in a new marking that is not in $\Delta(N)$, is the minimally restrictive LESP for $N(\mathbf{m}^0)$ [3].

A supervisory policy $\mathcal{P} : \mathcal{N}^n \times T \rightarrow \{0, 1\}$ is a *marking monotone policy* (MM-policy) if $\forall \widehat{\mathbf{m}} \geq \mathbf{m}, \forall t \in T, \mathcal{P}(\widehat{\mathbf{m}}, t) \geq \mathcal{P}(\mathbf{m}, t)$. That is, if a transition is permitted by an MM-policy at a marking, it will be permitted at a larger marking as well. If an MM-policy that is an LESP for $N(\mathbf{m}^0)$, is also an LESP for $N(\widehat{\mathbf{m}}^0), \forall \widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$, then it is said to be a *marking-monotone LESP* (MM-LESP) for $N(\mathbf{m}^0)$. The set

$$\Delta_M(N) = \{\mathbf{m}^0 : \exists \text{ an MM-LESP for } N(\mathbf{m}^0)\}$$

denotes the set of initial marking for there is an MM-LESP for the PN structure N. It follows that $\Delta_M(N) \subseteq \Delta(N)$.

A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \geq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 \in \mathcal{M}))$. A right-closed set, $\mathcal{M}$, is uniquely identified by its finite set of minimal elements denoted by $\min(\mathcal{M})$. The empty-set is right-closed by definition; and $\Delta_M(N) \subseteq \Delta(N)$, is right-closed for any PN structure N.

The $\mathcal{H}$-class of PN structures is identified by the following structural properties: (1) for each place, the weights associated with the outgoing arcs that terminate on uncontrollable transitions must be the smallest of all outgoing arc-weights; (2) the set of input places to each uncontrollable transition is no larger than the set of input places of any transition which shares a common input place with it. Formally stated, let $\Omega(t) = \{\widehat{t} \in T | {}^\bullet t \cap {}^\bullet \widehat{t} \neq \emptyset\}$ denote the set of transitions that share a common input place with $t \in T$ for a PN structure $N = (\Pi, T, \Phi, F)$. A PN structure $N \in \mathcal{H}$ if and only if $\forall p \in \Pi$, $\forall t_u \in p^\bullet \cap T_u$, we have $(\Gamma(p, t_u) = min_{t \in p^\bullet} \Gamma(p, t))$ $\wedge$ $(\forall t \in \Omega(t_u), {}^\bullet t_u \subseteq {}^\bullet t)$. For these classes of PNs, $\Delta(N)$ is *right-closed* [7].

III. Right-Closure of $\Delta(N)$

In Section I, we noted that the $\mathcal{H}$-class of PN structures is the largest among the classes identified in [5]–[8] for which the existence of an LESP is decidable and for which $\Delta(N)$ is right-closed. Consider the PN structure N_1 shown in Fig. 1. It does not belong to $\mathcal{H}$-class as the outgoing arcs of place p_1 violate the $\mathcal{H}$-class restriction. However, it can be verified that $\Delta(N_1) = \{\mathbf{m} \in \mathcal{N}^5 : (\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_4) + \mathbf{m}(p_5) \geq 1)\}$, is indeed right-closed. This example illustrates that there are PN structures that do not belong to $\mathcal{H}$-class but still have a right-closed $\Delta(N)$. In this section, we present a necessary and sufficient condition for the right-closure of $\Delta(N)$ for an arbitrary PN structure N.

Recall that for an uncontrollable transition t_u, $\mathbf{IN}_{t_u}$ is the smallest integer-valued vector that state-enables t_u. Let $\mathbf{P} = Int(conv(\{\mathbf{IN}_{t_u}\}_{t_u \in T_u}))$ (resp. $k \times \mathbf{P} = Int(conv(\{k \times \mathbf{IN}_{t_u}\}_{t_u \in T_u})), k \in \mathcal{N}$) denote the set of integer-valued vectors in the convex-hull of the columns of the input matrix $\mathbf{IN}$ (resp. k times the columns of the input matrix $\mathbf{IN}$) that correspond to the uncontrollable transitions in N.

Let $\widehat{\mathcal{H}}$ be a class of PN structures where for any $N \in \widehat{\mathcal{H}}$,

$$(\mathbf{m} \in \Delta(N)) \Rightarrow ((\mathbf{m} + \mathbf{P}) \subset \Delta(N)). \tag{2}$$

That is, if $\mathbf{m} \in \Delta(N)$, then $\forall \mathbf{x} \in \mathbf{P}, (\mathbf{m} + \mathbf{x}) \in \Delta(N)$. The operator "+" in Equation 2 denotes the Minkowski Sum as defined in Section II. Note that recursing over the expression in Equation 2 will give us an equivalent condition: $(\mathbf{m} \in \Delta(N)) \Rightarrow ((\mathbf{m} + k \times \mathbf{P}) \subset \Delta(N)), k \in \mathcal{N}$. The following result shows that for any $N \in \widehat{\mathcal{H}}$ the set $\Delta(N)$ is right-closed; and if $\Delta(N)$ right-closed, then $N \in \widehat{\mathcal{H}}$.

Theorem 1. *$(N \in \widehat{H}) \Leftrightarrow (\Delta(N)$ is right-closed).*

Proof. ($\Rightarrow$) If $\Delta(N) = \emptyset$, it is right-closed by definition. If $\Delta(N) \neq \emptyset$, we establish the result by proving the contrapositive. Assume $\Delta(N)$ is not right-closed. Particularly, assume there exists $\mathbf{m}^1 \in \Delta(N)$ such that $(\mathbf{m}^1 + \widehat{\mathbf{m}}) \notin \Delta(N)$. Now, $\Delta(N)$ for a fully controllable PN is right-closed. Therefore, if $(\mathbf{m}^1 + \widehat{\mathbf{m}}) \notin \Delta(N)$, then the set of uncontrollable transitions of N will be non-empty, and hence $\mathbf{P} = Int(conv(\{\mathbf{IN}_{t_u}\}_{t_u \in T_u}))$ is a non-empty set. Consider $\mathbf{m}^1 \in \Delta(N)$ and let Π_c denote the set of places connected to only controllable transitions (i.e. $\Pi_c^\bullet \cap T_u = \emptyset$). The initial token load of all $p \in \Pi_c$ can be increased to an arbitrarily large value and the initial marking will still be inside $\Delta(N)$. This is true because the supervisory policy can act as if the extra tokens in all $p \in \Pi_c$ never existed, and enforce liveness in the same way as for $\mathbf{m}^1$. Therefore, without loss of generality we can assume that the marking $(\mathbf{m}^1 + \widehat{\mathbf{m}}) \notin \Delta(N)$ has additional tokens in only those places that are connected to at least one uncontrollable transition. This implies, as $\mathbf{P}$ is the convex hull of the columns of the input matrix that correspond to the uncontrollable transitions, that there exists an integer k such that $(\mathbf{m}^1 + \widehat{\mathbf{m}}) \in (\mathbf{m}^1 + k \times \mathbf{P})$. On the other hand, we have $(\mathbf{m}^1 + \widehat{\mathbf{m}}) \notin \Delta(N)$. Then by the characterization of $\widehat{H}$ class above, we have $N \notin \widehat{H}$.

($\Leftarrow$) We prove this via the contrapositive. Assume $N \notin \widehat{H}$. This means that there exists an $\mathbf{m} \in \Delta(N)$ for which there exists a (larger) marking inside the set $\mathbf{m} + \mathbf{P}$ at which the PN is not live. This implies $\Delta(N)$ is not right-closed.

□

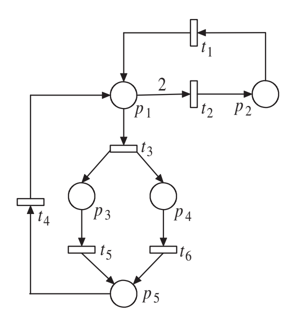


Fig. 1: A PN structure $N_1 \notin \mathcal{H}$.

Coming back to the Petri Net N_1 in Figure 1, the set $\mathbf{P}$ for N_1 consists of five vectors of $\mathcal{N}^5$, viz., $\{(1\ 0\ 0\ 0\ 0)^T, (0\ 2\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$. For any marking $\mathbf{m} \in \Delta(N_1)$ (i.e. $\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_4) + \mathbf{m}(p_5) \geq 1$), it is easy to verify that $\mathbf{m}^* \in \mathbf{m} + \mathbf{P}$ satisfies $\mathbf{m}^*(p_1) + \mathbf{m}^*(p_2) + \mathbf{m}^*(p_3) + \mathbf{m}^*(p_4) + \mathbf{m}^*(p_5) \geq 1$. Thus, $N_1 \in \widehat{\mathcal{H}}$.

In Section V, we prove that the necessary and sufficient condition of Theorem 1 cannot be tested for an arbitrary PN structure. To establish this, we need the results presented in the next section where we consider the decidability of "*Is* $\Delta(N) = \emptyset$*?*" and "*Is* $\Delta(N) \neq \emptyset$*?*" for arbitrary PN structures.

IV. "*Is* $\Delta(N) = \emptyset$?" and "*Is* $\Delta(N) \neq \emptyset$?" are not Semi-Decidable

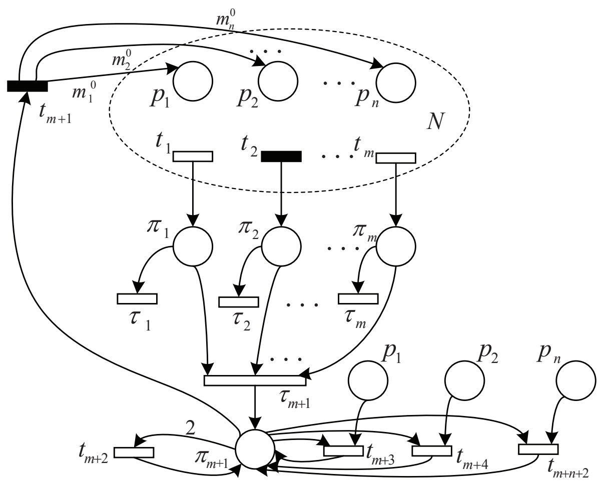


Fig. 2: The PN structure $\widetilde{N} = (\widetilde{\Pi}, \widetilde{T}, \widetilde{\Phi}, \widetilde{\Gamma})$ used for deciding "*Is* $\Delta(N) = \emptyset$?".

From an arbitrary PN structure $N = (\Pi, T, \Phi, \Gamma)$, we construct $\widetilde{N} = (\widetilde{\Pi}, \widetilde{T}, \widetilde{\Phi}, \widetilde{\Gamma})$ as follows:

1) Create $m+1$ places such that $\widetilde{\Pi} = \Pi \cup \{\pi_i\}_{i=1}^{m+1}$;
2) Create $n+2$ transitions such that $\widetilde{T} \leftarrow T \cup \{t_{m+i}\}_{i=1}^{n+2}$, where with the exception of t_{m+1}, all other newly added transitions are uncontrollable;
3) Create $m+1$ uncontrollable transitions: $\widetilde{T} \leftarrow \widetilde{T} \cup \{\tau_i\}_{i=1}^{m+1}$;
4) The arcs are:

$$\begin{aligned}\widetilde{\Phi} \leftarrow & \Phi \cup \{(t_{m+1}, p_i)\}_{i=1}^{n} \cup \{(t_i, \pi_i), (\pi_i, \tau_i), (\pi_i, \tau_{m+1})\}_{i=1}^{m} \\ & \cup \{(p_i, t_{m+2+i}), (\pi_{m+1}, t_{m+2+i}), (t_{m+2+i}, \pi_{m+1})\}_{i=1}^{n} \\ & \cup \{(t_{m+2}, \pi_{m+1}), (\pi_{m+1}, t_{m+2}), (\pi_{m+1}, t_{m+1})\} \\ & \cup \{(\tau_{m+1}, \pi_{m+1})\};\end{aligned}$$

5) The arc weights are: $\{\widetilde{\Gamma}((t_{m+1}, p_i) = m_i^0)\}_{i=1}^{n}$, $\widetilde{\Gamma}(\pi_{m+1}, t_{m+2}) = 2$. All other weights for the newly added arcs are unitary.

The PN structure $\widetilde{N} = (\widetilde{\Pi}, \widetilde{T}, \widetilde{\Phi}, \widetilde{\Gamma})$ that results from this construction is shown in Fig. 2. N is an arbitrary PN and its structure is not drawn in the figure. The places $\{p_i\}_{i=1}^{n}$ and transitions $\{t_i\}_{i=1}^{m}$ denote the places and transitions of N.

Recall from Section II that a transition t_k is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$. A policy $\mathcal{P}$ is a *liveness enforcing supervisory policy* (LESP) for $N(\mathbf{m}^0)$ if all transitions in $N(\mathbf{m}^0)$ are live under $\mathcal{P}$.

Let $\widetilde{\mathbf{m}}^0$ be an initial marking of $\widetilde{N}$. Transition τ_{m+1} is live *if and only if* (iff) a marking that places at least a token in each of the places π_i, for all $i \in \{1, 2, \ldots, m\}$, is reachable from any marking that is reachable from $\widetilde{\mathbf{m}}^0$. Firing of the uncontrollable transition τ_i can empty the tokens in π_i, for all $i \in \{1, 2, \ldots, m\}$. Therefore, τ_{m+1} is live iff the token load of places π_i for all $i \in \{1, 2, \ldots, m\}$ can be replenished as often as necessary. Since transition t_i is the input transition of the place π_i for all $i \in \{1, 2, \ldots, m\}$, τ_{m+1} is live iff PN N can be made live. More formally, if $\mathbf{m}^0$ is an initial marking of N, then τ_{m+1} is live iff there exists a supervisory policy $\mathcal{P}$ such that $\forall t_i \in T, \forall \mathbf{m}^k \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^k, \mathcal{P})$ such that $t_i \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t) = 1$. This observation can also be restated as: a marking that places one (or arbitrarily large number of tokens) token in π_{m+1} is reachable from any marking that is reachable from the initial marking iff $N(\mathbf{m}^0)$ can be made live by supervision.

Consider the place π_{m+1} and assume for the sake of discussion that t_{m+1} is control-disabled. If π_{m+1} has more than one token, the uncontrollable transition t_{m+2} can fire repeatedly till there is just one token in π_{m+1}. That is, if a policy disables t_{m+1}, then a marking at which π_{m+1} has 1 token is always reachable from a marking at which π_{m+1} has $k > 0$ tokens.

Besides, if π_{m+1} has a non-zero token load, then the places $\{p_1, p_2, \ldots, p_n\}$ in PN N can be emptied through an appropriate number of firings of members of the uncontrollable transition set $\{t_{m+3}, t_{m+4}, \ldots, t_{m+n+2}\}$. In other words, if a policy disables t_{m+1} at marking $\mathbf{m}$ of $\widetilde{N}$ for which $\mathbf{m}(\pi_{m+1}) \neq 0$ and $\mathbf{m}(p_i) \neq 0$ for some $i \in \{1, \ldots, n\}$, then a marking at which the places $\{p_1, p_2, \ldots, p_n\}$ are all empty is reachable from $\mathbf{m}$.

We use $\overline{\mathbf{m}} \in \mathcal{N}^{card(\widetilde{\Pi})}$ to represent this marking of $\widetilde{N}$ at which π_{m+1} has one token while all other places have zero tokens in them. We use the ideas from the preceding two paragraphs to synthesize a policy which does not control-enable transition t_{m+1} until the PN reaches the marking $\overline{\mathbf{m}}$. At $\overline{\mathbf{m}}$, the firing of the transition t_{m+1} places m_i^0-many tokens in place p_i, where $i \in \{1, 2, \ldots, n\}$. This is akin to initializing the PN structure N with a marking $\mathbf{m}^0$, while the rest of the places of $\widetilde{N}$ are all empty. Here we have used $\mathbf{m}^0$ to denote the marking for which $\mathbf{m}^0(p_i) = m_i^0$. Following the discussion above, a token (or arbitrarily large number of tokens) is guaranteed to be added to place π_{m+1} if and only if $\mathbf{m}^0 \in \Delta(N)$. Once there is a token in π_{m+1}, transition t_{m+1} cannot be control-enabled until the PN reaches the marking $\overline{\mathbf{m}}$, and the sequence can be repeated, making $\widetilde{N}$ live. This is the main idea of the proofs below.

Observation 1. $(\Delta(\widetilde{N}) \neq \emptyset) \Leftrightarrow (\overline{\mathbf{m}} \in \Delta(\widetilde{N}))$

Proof. ($\Rightarrow$) If there is a marking $\mathbf{m}^1 \in \Delta(\widetilde{N})$, then following the introductory discussion above, there is a marking $\mathbf{m}^2 \in \Delta(\widetilde{N})$ reachable from $\mathbf{m}^1$ under the supervision of any LESP for $\widetilde{N}(\mathbf{m}^1)$, where $\mathbf{m}^2(\pi_{m+1}) \neq 0$. Additionally, $\exists \sigma_u \in (\{t_{m+2}, t_{m+3}, \ldots, t_{m+n+2}\} \cup \{\tau_1, \tau_2, \ldots, \tau_m\})^*$ (note, σ_u is string of uncontrollable transitions) such that $\mathbf{m}^2 \xrightarrow{\sigma_u} \overline{\mathbf{m}}$. That is, $\overline{\mathbf{m}}$ is reachable from $\mathbf{m}^2$. Since $\mathbf{m}^2 \in \Delta(\widetilde{N})$, and $\mathbf{m}^2 \xrightarrow{\sigma_u} \overline{\mathbf{m}}$, where σ_u is a string of uncontrollable transitions, by control invariance, it follows that $\overline{\mathbf{m}} \in \Delta(\widetilde{N})$.

($\Leftarrow$) If $\overline{\mathbf{m}} \in \Delta(\widetilde{N})$ then $\Delta(\widetilde{N}) \neq \emptyset$ by definition. □

Observation 2. $(\overline{\mathbf{m}} \in \Delta(\widetilde{N})) \Leftrightarrow (\mathbf{m}^0 \in \Delta(N))$

Proof. ($\Rightarrow$) If $\overline{\mathbf{m}} \in \Delta(\widetilde{N})$, then since $T_e(\widetilde{N}, \overline{\mathbf{m}}) = \{t_{m+1}\}$, we have $\overline{\mathbf{m}} \xrightarrow{t_{m+1}} \overline{\mathbf{m}}^1$ under the supervision of any LESP for $\widetilde{N}(\overline{\mathbf{m}})$. At $\overline{\mathbf{m}}^1$, the PN structure N is initialized with a marking $\mathbf{m}^0$, while the rest of the places of $\widetilde{N}$ are all empty. Since $\overline{\mathbf{m}}^1 \in \Delta(\widetilde{N})$ it follows that $\mathbf{m}^0 \in \Delta(N)$. If it were otherwise, the transition τ_{m+1} cannot be made live in $\widetilde{N}(\overline{\mathbf{m}}^1)$, and we must conclude that $\overline{\mathbf{m}}^1 \notin \Delta(\widetilde{N})$.

($\Leftarrow$) If $\mathbf{m}^0 \in \Delta(N)$, there is an LESP $\mathcal{P}$ for $N(\mathbf{m}^0)$. This LESP is used to construct an LESP $\widetilde{\mathcal{P}}$ for $\widetilde{N}(\overline{\mathbf{m}})$ as follows for $\underline{\mathbf{m}} \in \mathcal{N}^{card(\widetilde{\Pi})}, \widetilde{t} \in \widetilde{T}$

$$\widetilde{\mathcal{P}}(\underline{\mathbf{m}}, \widetilde{t}) = \begin{cases} \mathcal{P}(\underline{\mathbf{m}}(\Pi), \widetilde{t}) & \text{if } \widetilde{t} \in T, \\ 1 & \text{if } \widetilde{t} \in (\widetilde{T} - T - \{t_{m+1}\}), \\ 1 & \text{iff } (\widetilde{t} = t_{m+1}) \wedge (\underline{\mathbf{m}} = \overline{\mathbf{m}}), \end{cases}$$

where, $\underline{\mathbf{m}}(\Pi)$ denotes the marking of the subnet N. The fact that $\widetilde{\mathcal{P}}$ is an LESP for $\widetilde{N}(\overline{\mathbf{m}})$ follows directly from the construction of $\widetilde{N}$ and the fact that $\mathcal{P}$ is an LESP for $N(\mathbf{m}^0)$. □

Theorem 2. *1) "Is $\Delta(N) \neq \emptyset$?" is not semi-decidable.*
2) "Is $\Delta(N) = \emptyset$?" is not semi-decidable.

Proof. By Observations 1 and 2, we have $(\Delta(\widetilde{N}) \neq \emptyset) \Leftrightarrow (\mathbf{m}^0 \in \Delta(N))$. This result follows directly from the fact that neither "*Is* $\mathbf{m}^0 \in \Delta(N)$*?*" nor "*Is* $\mathbf{m}^0 \notin \Delta(N)$*?*" is semi-decidable [5]. □

In the next section, we use Theorem 2 to prove that "*Is* $\Delta(N)$ *right-closed?*" is not decidable.

V. "*Is* $\Delta(N)$ *right-closed?*" is not decidable

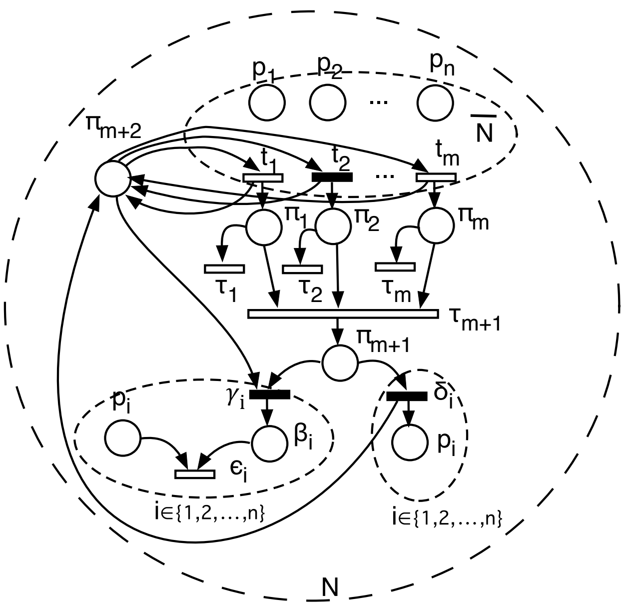


Fig. 3: The PN structure $N = (\Pi, T, \Phi, \Gamma)$ used for deciding "*Is* $\Delta(N)$ *right-closed?*".

In this section, we use the fact that $\Delta(N) = \emptyset$ is right-closed to prove that "*Is* $\Delta(N)$ *right-closed?*" is not decidable. We construct a partially controlled PN $N = (\Pi, T, \Phi, \Gamma)$ from an arbitrary partially controlled PN $\overline{N} = (\Pi_1, T_1, \Phi_1, \Gamma_1)$ as follows:

1) Create $m+n+2$ places such that $\Pi = \Pi_1 \cup \{\pi_i\}_{i=1}^{m+2} \cup \{\beta_i\}_{i=1}^n$.
2) Create $3n+m+1$ transitions: $T = T_1 \cup \{\tau_i\}_{i=1}^{m+1} \cup \{\gamma_i\}_{i=1}^n \cup \{\epsilon_i\}_{i=1}^n \cup \{\delta_i\}_{i=1}^n$; where $\{\gamma_i\}_{i=1}^n$ and $\{\delta_i\}_{i=1}^n$ are controllable transitions, and $\{\tau_i\}_{i=1}^{m+1}$ and $\{\epsilon_i\}_{i=1}^n$ are uncontrollable transitions.
3) The arcs are:

$$\begin{aligned}\Phi_1 = &\Phi \cup \{(t_i, \pi_i), (\pi_i, \tau_i), (\pi_i, \tau_{m+1}), (\pi_{m+2}, t_i), (t_i, \pi_{m+2})\}_{i=1}^m \\ &\cup \{(\pi_{m+2}, \gamma_i), (\gamma_i, \beta_i), (\beta_i, \epsilon_i), (p_i, \epsilon_i), (\delta_i, p_i)\}_{i=1}^n \\ &\cup \{(\pi_{m+1}, \gamma_i), (\pi_{m+1}, \delta_i), (\delta_i, \pi_{m+2})\}_{i=1}^n.\end{aligned}$$

4) Weights for the newly added arcs are unitary.

The construction can be divided into five parts:

1) An arbitrary net $\overline{N}$, which is the core of the construction. Places $\{p_i\}_{i=1}^n$ and transitions $\{t_i\}_{i=1}^m$ belong to PN $\overline{N}$ whose structure is not drawn in the construction.
2) The *enable place* π_{m+2} which is required to have a non-zero token load if any transition in $\overline{N}$ is to be state-enabled.
3) Places $\{\pi_i\}_{i=1}^m$ and transitions $\{\tau_i\}_{i=1}^m$ capture the liveness property of subnet $\overline{N}$ as described in the introductory discussion in Section IV.
4) Places $\{\beta_i\}_{i=1}^n$ and transitions $\{\gamma_i, \epsilon_i\}_{i=1}^n$: Each time a (controllable) γ_i-transition is permitted to fire, it decreases (resp. increments) the token load of its input-place set (resp. output-place set) $\{\pi_{m+1}, \pi_{m+2}\}$ (resp. $\{\beta_i\}$). The subsequent firing of the (uncontrollable) ϵ_i-transition decrements the number of tokens in place p_i from $\overline{N}$ by unity.
5) Transitions $\{\delta_i\}_{i=1}^n$: The firing of a (controllable) δ_i-transition increments the token load of place p_i. It also replenishes the tokens in π_{m+2}. In essence, the firing of a δ_i-transition cancels the effect of permitting a γ_i-transition (cf. Item 4) on place π_{m+2}, and the effect of firing of ϵ_i-transition on place p_i. Since the transitions $\{\gamma_i, \delta_i\}_{i=1}^n$ are controllable, the supervisory policy can select which one of them is to be control-enabled at any marking.

The observation that is key to the decidability result in this section is that there is a marking in $\Delta(N)$ if and only if there is a marking in $\Delta(\overline{N})$. The main idea is as follows. Assume there exists a marking $\mathbf{m}^1 \in \Delta(\overline{N})$. Let us use $\overline{\mathbf{m}}^1$ to denote the marking of N that initializes $\overline{N}$ under $\mathbf{m}^1$, with a single token in π_{m+2}, and zero tokens elsewhere. We argue that $(\mathbf{m}^1 \in \Delta(\overline{N})) \Rightarrow (\overline{\mathbf{m}}^1 \in \Delta(N))$. Now, starting at $\overline{\mathbf{m}}^1$ the transitions in T_1 can be made live under supervision as $\mathbf{m}^1 \in \Delta(\overline{N})$. This ensures that the markings for which the place π_{m+1} has arbitrarily large number of tokens are reachable from any marking that is reachable from $\overline{\mathbf{m}}^1$. For illustration, let us use $\overline{\mathbf{m}}^j$ to denote one such marking.

1) At $\overline{\mathbf{m}}^j$, place π_{m+1} has two tokens, π_{m+2} has one token, $\overline{\mathbf{m}}^j(\Pi_1) \in \Delta(\overline{N})$ and all other places have zero tokens. As discussed earlier, $\overline{\mathbf{m}}^j$ is reachable from $\overline{\mathbf{m}}^1$. At $\overline{\mathbf{m}}^j$, pick any p_i that has a non-zero token load. The corresponding controllable transition γ_i is state- and control-enabled at this marking. In fact, since $\mathbf{m}^1 \in \Delta(\overline{N})$, markings for which γ_i is state- and control-enabled are reachable from $\overline{\mathbf{m}}^j$ for every $i \in \{1, \ldots n\}$.
2) We have $\gamma_i^\bullet = \beta_i$ and ${}^\bullet\epsilon_i = \{p_i, \beta_i\}$. The firing of γ_i will remove a token each from π_{m+1} and π_{m+2} and add one token to place β_i. Since π_{m+2} had only one token, none of the transitions in T_1 can fire and the marking of $\overline{N}$ cannot change. Thus, a marking that state-enables the uncontrollable transition ϵ_i is reachable from $\overline{\mathbf{m}}^j$. In fact,

since $\gamma_i^\bullet = \beta_i$ and $^\bullet\epsilon_i = \{p_i, \beta_i\}$, following the discussion in Item 1 above, markings for which ϵ_i is state-enabled are also reachable from $\overline{\mathbf{m}}^j$ for every $i \in \{1, \dots n\}$. The firing of ϵ_i will decrease the token-load of p_i and β_i by one.

3) Following this, the corresponding transition δ_i is control-enabled. The firing of δ_i replenishes the token-load of p_i and π_{m+2} by one, effectively cancelling the effect of the firing of γ_i and ϵ_i on them, as discussed above.
4) Since $\mathbf{m}^1 \in \Delta(\overline{N})$, the tokens in place π_{m+1} can be replenished as often as necessary and the whole process can be repeated for each γ_i, ϵ_i and δ_i, for all $i \in \{1, 2, \dots, n\}$. Thus, markings that state- and control-enable each of the transitions in N are reachable from every marking that is reachable from the initial marking ($\overline{\mathbf{m}}^1$). All transitions in $N(\overline{\mathbf{m}}^1)$ are live under supervision, and $\Delta(N) \neq \emptyset$.

We formally define an LESP in the proof of Observation 3 that enables the controllable transitions in the sequence discussed in above items.

Observation 3. $(\Delta(\overline{N}) \neq \emptyset) \Leftrightarrow (\Delta(N) \neq \emptyset)$

Proof. ($\Rightarrow$) Assume there exists a marking $\mathbf{m}^1 \in \Delta(\overline{N})$. Consider another marking $\overline{\mathbf{m}}^1$ of the PN N that initializes (a) the places of $\overline{N}$ (i.e. $\{p_1, \dots p_n\}$) with token loads identified by the marking $\mathbf{m}^1$, that is $\overline{\mathbf{m}}^1(\Pi_1) = \mathbf{m}^1$, (b) a single token in π_{m+2} and (c) zero tokens in all other places. We show that $\overline{\mathbf{m}}^1 \in \Delta(N)$ by constructing an LESP $\mathcal{P}$ for $N(\overline{\mathbf{m}}^1)$.

Let $\mathcal{P}$ be a policy such that $\forall t_c \in T_1$:

$$(\mathcal{P}(\overline{\mathbf{m}}^2, t_c) = 0) \Leftrightarrow ((\overline{\mathbf{m}}^2 \xrightarrow{t_c} \overline{\mathbf{m}}^3) \wedge (\overline{\mathbf{m}}^3(\Pi_1) \notin \Delta(\overline{N}))) \qquad (3)$$

That is, it prevents a controllable transition in T_1 if and only if its firing takes the marking of $\overline{N}$ outside $\Delta(\overline{N})$. Since $\mathbf{m}^1 \in \Delta(\overline{N})$, all transitions in T_1 are live when $N(\overline{\mathbf{m}}^1)$ is under the supervision of $\mathcal{P}$. Consequently, from the definition of liveness, $\forall k \in \mathcal{N}, \forall \overline{\mathbf{m}}^4 \in \Re(N, \overline{\mathbf{m}}^1, \mathcal{P}), \exists \overline{\mathbf{m}}^5 \in \Re(N, \overline{\mathbf{m}}^4, \mathcal{P})$ such that $\overline{\mathbf{m}}^5(\pi_{m+1}) \geq k$, $\overline{\mathbf{m}}^5(\Pi_1) \in \Delta(\overline{N})$, and $\overline{\mathbf{m}}^5(\pi_{m+2}) = 1$.

The supervisory policy $\mathcal{P}$ control-enables a transition γ_i, where $i \in \{1, \dots, n\}$, at a marking $\overline{\mathbf{m}}^6 \in \Re(N, \overline{\mathbf{m}}^1, \mathcal{P})$ if and only if (a) $\overline{\mathbf{m}}^1 \xrightarrow{\sigma} \overline{\mathbf{m}}^6$ under the supervision of $\mathcal{P}$, and $\#(\sigma, \gamma_i) = \#(\sigma, \delta_i)$, (b) $\overline{\mathbf{m}}^6(p_i) \neq 0$, (c) $\overline{\mathbf{m}}^6(\pi_{m+2}) = 1$, and (d) $\overline{\mathbf{m}}^6(\pi_{m+1}) \geq 2$. That is, if $\overline{\mathbf{m}}^6 \xrightarrow{\gamma_i} \overline{\mathbf{m}}^7$ under the supervision of $\mathcal{P}$, then $T_e(N, \overline{\mathbf{m}}^7) = \{\epsilon_i\}$ and $\overline{\mathbf{m}}^7(\pi_{m+2}) = 0$. Here we use the notation $\#(\sigma, t)$ to denote the number of occurrences of transition t in a valid firing string σ.

A transition in the set $\{\delta_i\}_{i=1}^n$ is control-enabled at $\overline{\mathbf{m}}^8 \in \Re(N, \overline{\mathbf{m}}^1, \mathcal{P})$ if and only if (i) $\overline{\mathbf{m}}^8(\pi_{m+2}) = 0$, and (ii) $\exists \overline{\mathbf{m}}^6 \in \Re(N, \overline{\mathbf{m}}^1, \mathcal{P})$ such that $\overline{\mathbf{m}}^6 \xrightarrow{\gamma_i \epsilon_i} \overline{\mathbf{m}}^8$ under the supervision of $\mathcal{P}$. That is, if $\overline{\mathbf{m}}^8 \xrightarrow{\delta_i} \overline{\mathbf{m}}^9$ under the supervision of $\mathcal{P}$, then $\overline{\mathbf{m}}^9(\pi_{m+2}) = 1$ and $\overline{\mathbf{m}}^9(\Pi_1) \in \Delta(\overline{N})$.

Thus, following the discussion in the paragraph preceding this observation, all transitions in $N(\overline{\mathbf{m}}^1)$ are live under supervision of $\mathcal{P}$, and $\Delta(N) \neq \emptyset$.
($\Leftarrow$) We prove this via the contrapositive. If $\Delta(\overline{N}) = \emptyset$, then τ_{m+1} cannot be made live, and $\Delta(N) = \emptyset$. □

Observation 4. $(\Delta(N) \neq \emptyset) \Leftrightarrow$ *($\Delta(N)$ is not right-closed)*

Proof. ($\Rightarrow$) Suppose $\Delta(N) \neq \emptyset$, consider the marking $\overline{\mathbf{m}}^1$ from Observation 3. Next consider a marking, $\overline{\mathbf{m}}^2 > \overline{\mathbf{m}}^1$, where $\overline{\mathbf{m}}^2(p) = \overline{\mathbf{m}}^1(p), \forall p \in (\Pi - \{\beta_i\}_{i=1}^n)$, and $\overline{\mathbf{m}}^2(\beta_i) \geq \overline{\mathbf{m}}^1(p_i)$, for each $p_i \in \Pi_1$. At the marking $\overline{\mathbf{m}}^2$, the uncontrollable transitions in the set $\{\epsilon_i\}_{i=1}^n$ can fire as often as necessary to empty all places in the set $\{p_i\}_{i=1}^n$. Consequently, there can be no LESP for $N(\overline{\mathbf{m}}^2)$, and $\overline{\mathbf{m}}^2 \notin \Delta(N)$ while $\overline{\mathbf{m}}^1 \in \Delta(N)$. Therefore, if $\Delta(N) \neq \emptyset$, it cannot be right-closed.

($\Leftarrow$) By definition, $(\Delta(N) = \emptyset) \Rightarrow (\Delta(N)$ is right-closed$)$. □

Theorem 3. *"Is $\Delta(N)$ right-closed?" is not decidable.*

Proof. By Observation 4, we have that $(\Delta(N) \neq \emptyset) \Leftrightarrow (\Delta(N)$ is not right-closed). This result follows directly from the fact that neither "*Is* $\Delta(N) = \emptyset$?" nor "*Is* $\Delta(N) \neq \emptyset$?" is semi-decidable (by Theorem 2). □

In this section, we proved that "*Is* $\Delta(N)$ *is right-closed*?" is not decidable. In the next section we consider the decidability of "*Is there a (non-empty) right-closed subset of* $\Delta(N)$*?*".

VI. "*Is there a right-closed subset of* $\Delta(N)$*?*" and "*Is there no right-closed subset of* $\Delta(N)$*?*" are not semi-decidable

In this section we look at procedures for finding right-closed subsets of $\Delta(N)$ for an arbitrary PN N. Every $\Delta(N)$, trivially, has the empty set as its right-closed subset. Therefore, we consider only the non-empty subsets of $\Delta(N)$. We use the construction in Fig. 2. Recall from Section IV that at the marking $\overline{\mathbf{m}}$, π_{m+1} has one token while all other places have zero tokens in them. In Observation 2, we noted that the supervisory policy that enforces liveness in $\widetilde{N}$ enables t_{m+1} only after $\widetilde{N}$ has reached the marking $\overline{\mathbf{m}}$. The marking $\overline{\mathbf{m}}$ is reachable from any marking larger than $\overline{\mathbf{m}}$ through the firing of uncontrollable transitions t_{m+2} to t_{m+n+2}. This observation forms the basis of the next result.

Observation 5. *Let* $\underline{\mathbf{m}} \geq \overline{\mathbf{m}}$, *then* $(\Delta(\widetilde{N}) \neq \emptyset) \Leftrightarrow (\underline{\mathbf{m}} \in \Delta(\widetilde{N}))$.

Proof. ($\Rightarrow$) By Observations 1 and 2, $\Delta(\widetilde{N}) \neq \emptyset$ if and only if $\overline{\mathbf{m}} \in \Delta(N)$. Assume $\widetilde{N}$ is initialized with the marking $\underline{\mathbf{m}} \geq \overline{\mathbf{m}}$. We define a supervisory policy $\widetilde{\mathcal{P}}^1$ as follows. For $\underline{\mathbf{m}} \in \mathcal{N}^{card(\widetilde{\Pi})}, \widetilde{t} \in \widetilde{T}$

$$\widetilde{\mathcal{P}}^1(\underline{\mathbf{m}}, \widetilde{t}) = \begin{cases} 1 & \text{if } \widetilde{t} \in (\widetilde{T} - \{t_{m+1}\}), \\ 0 & \text{iff } (\widetilde{t} = t_{m+1}) \wedge (\underline{\mathbf{m}} \neq \overline{\mathbf{m}}), \\ \widetilde{\mathcal{P}}(\overline{\mathbf{m}}, \widetilde{t}) & \text{iff } (\widetilde{t} = t_{m+1}) \wedge (\underline{\mathbf{m}} = \overline{\mathbf{m}}). \end{cases}$$

The marking $\overline{\mathbf{m}}$ is reachable from all markings in $\Re(\widetilde{N}, \underline{\mathbf{m}}, \widetilde{\mathcal{P}}^1)$, and $\overline{\mathbf{m}} \in \Delta(\widetilde{N})$. Once the PN reaches the marking $\overline{\mathbf{m}}$, we switch to the supervisory policy $\widetilde{\mathcal{P}}$ as defined in Observation 2. Therefore, $\forall t \in \widetilde{T}, \forall \mathbf{m}^i \in \Re(N, \underline{\mathbf{m}}, \widetilde{\mathcal{P}}^1), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \widetilde{\mathcal{P}}^1)$ such that $t \in T_e(N, \mathbf{m}^j)$ and $\widetilde{\mathcal{P}}^1(\mathbf{m}^j, t) = 1$. Thus, $\underline{\mathbf{m}} \in \Delta(\widetilde{N})$.

($\Leftarrow$) Straightforward by definition. □

Observation 6. $(\Delta(\widetilde{N}) \neq \emptyset) \Leftrightarrow$ *($\exists \mathcal{M} \subseteq \Delta(\widetilde{N})$, such that $\mathcal{M}$ is right-closed, and* $\min(\mathcal{M}) = \{\overline{\mathbf{m}}\}$*).*

This observation follows directly from Observation 5. The next result follows from Observation 6 and Theorem 2.

Theorem 4. *For an arbitrary PN N,*

1) *"Is there a right-closed subset of* $\Delta(N)$*?" is not semi-decidable.*
2) *"Is there no right-closed subset of* $\Delta(N)$*?" is not semi-decidable.*

In this section we proved that determining if there is a right-closed subset of $\Delta(N)$ for an arbitrary PN N is not decidable. In the next section, we identify the $\mathcal{K}$-class of PN structures, which to the best of our knowledge, is the largest characterized class of PNs for which $\Delta(N)$ is right-closed.

VII. An extension of the $\mathcal{H}$-class of PNs for which Right-closure of $\Delta(N)$ is testable

Recall the definition of $\mathcal{H}$-class of PNs introduced earlier in Section II. Let, $\Omega(t) = \{\widehat{t} \in T | {}^{\bullet}t \cap {}^{\bullet}\widehat{t} \neq \emptyset\}$, denote the set of transitions that share a common input place with $t \in T$ for a PN structure $N = (\Pi, T, \Phi, \Gamma)$. $N \in \mathcal{H}$ if and only if $\forall p \in \Pi, \forall t_u \in p^{\bullet} \cap T_u$:

$$\Gamma(p, t_u) = (min_{t \in p^{\bullet}} \Gamma(p, t)) \wedge (\forall t \in \Omega(t_u), {}^{\bullet} t_u \subseteq {}^{\bullet} t) \tag{4}$$

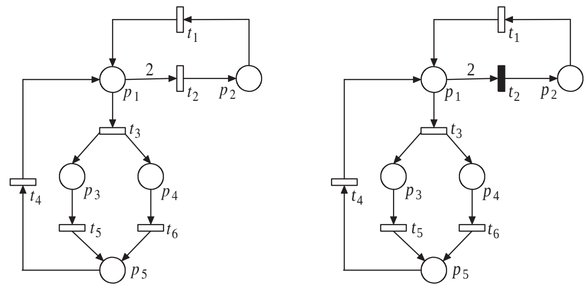
However, there are PNs that do not belong to $\mathcal{H}$-class but still have a right-closed $\Delta(N)$. As discussed at the beginning of Section III, the PN N_1 of Figure 1, shown again in Figure 4a, is one such example. We use this example as a motivation for the procedure to extend the $\mathcal{H}$-class of PNs.

For a given PN N, let $\mathscr{T}$ denote the collection of uncontrollable transitions which violate the condition in Equation 4. We construct a PN $N^* = (\Pi^*, T^*, \Phi^*, \Gamma^*)$ from N by changing those uncontrollable transitions that violate the condition of Equation 4, to controllable transitions. That is, $\Pi^* = \Pi, \Phi^* = \Phi, \Gamma^* = \Gamma, T^* = T_c^* \cup T_u^*, T_c^* = T_c \cup \mathscr{T}$, and $T_u^* = T_u - \mathscr{T}$. The resulting PN $N^* \in \mathcal{H}$ by construction, and therefore $\Delta(N^*)$ is right-closed. We also have $\Delta(N) \subseteq \Delta(N^*)$ as N^* has more controllable transitions as compared to N. Note that for $N \in \mathcal{H}$, $\mathscr{T}$ is empty.

Observation 7. *For an arbitrary PN N, if* $\forall t \in \mathscr{T}, \forall \mathbf{m} \in \min(\Delta(N^*)), \exists \mathbf{m}' \in \min(\Delta(N^*))$*, such that* $\max\{\mathbf{m}, \mathbf{IN}_t\} + \mathbf{C} \cdot \mathbf{1}_t \geq \mathbf{m}'$*, then* $\Delta(N^*) = \Delta(N)$.

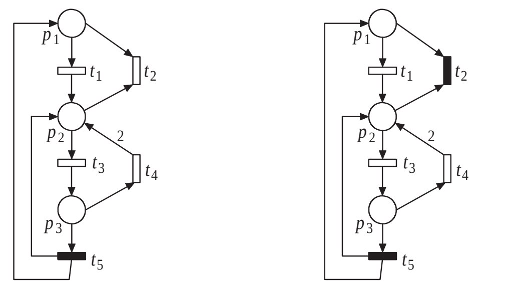
Proof. For a marking $\mathbf{m}$, the marking $\max\{\mathbf{m}, \mathbf{IN}_t\}$ is the smallest marking greater than or equal to $\mathbf{m}$ which state-enables transition t. If $\forall \mathbf{m} \in \min(\Delta(N^*))$, $\exists \mathbf{m}' \in \min(\Delta(N^*))$ such that $\max\{\mathbf{m}, \mathbf{IN}_t\} + \mathbf{C} \cdot \mathbf{1}_t \geq \mathbf{m}'$, then the firing of t at $\max\{\mathbf{m}, \mathbf{IN}_t\}$ will result in a marking that is in $\Delta(N^*)$. It follows that firing of t from any marking larger than $\max\{\mathbf{m}, \mathbf{IN}_t\}$ will also result in a marking in $\Delta(N^*)$ (as $\Delta(N^*)$ is right-closed). Therefore, the minimally restrictive LESP for $N^*(\mathbf{m}^0)$ will control enable t at every marking in $\Delta(N^*)$ and t can effectively be considered as an uncontrollable transition under its supervision. Consequently, the minimally restrictive LESP for $N^*(\mathbf{m}^0)$ also enforces liveness on $N(\mathbf{m}^0)$. Thus, we have $(\mathbf{m}^0 \in \Delta(N^*)) \Rightarrow (\mathbf{m}^0 \in \Delta(N))$ and $\Delta(N^*) \subseteq \Delta(N)$. Since N^* has more controllable transitions as compared to N, we already have that $\Delta(N) \subseteq \Delta(N^*)$. Therefore, $\Delta(N^*) = \Delta(N)$. □

It follows from Observation 7 that $\Delta(N^*) = \Delta(N)$ if and only if $\Delta(N^*)$ is control invariant with respect to all $t \in \mathscr{T}$. Algorithm 1 uses this observation to define a more general



(a) PN $N_1 \in \mathcal{K}$ (b) Transformed PN N_1^*

Fig. 4: The PN $N_1 \in \mathcal{K}$ and N_1^*. We have $\min(\Delta(N_1)) = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 1\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$.



(a) PN $N_2 \notin \mathcal{K}$ (b) Transformed PN N_2^*

Fig. 5: The PN $N_2 \notin \mathcal{K}$ and N_2^*. $\Delta(N_3) = \{((\mathbf{m}(p_1) - \mathbf{m}(p_2))_{mod2} = 1) \vee (\mathbf{m}(p_2) \geq \mathbf{m}(p_1)) \vee (\mathbf{m}(p_3) \geq 1)\}$ is not right-closed.

class of PNs, the class $\mathcal{K}$, for which $\Delta(N) = \Delta(N^*)$. It takes in the PN structure N as input and outputs if $N \in \mathcal{K}$ or not.

Algorithm 1 IsNinKClass?(N)

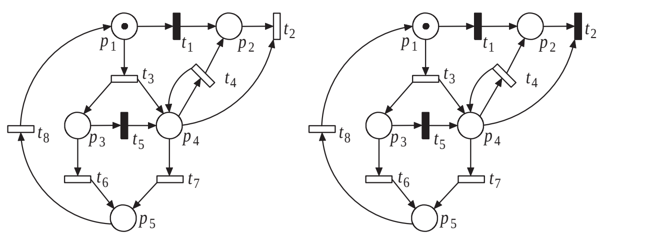
1: Construct $N^* = (\Pi^*, T^*, \Phi^*, \Gamma^*)$ and calculate $\Delta(N^*)$.
2: Change all $t \in \mathscr{T}$ to uncontrollable transitions. If $\Delta(N^*)$ is control invariant with respect to all $t \in \mathscr{T}$, then $N \in \mathcal{K}$.

We have the following inclusion relation: $\mathcal{H} \subset \mathcal{K} \subset \widehat{\mathcal{H}}$, where $\widehat{\mathcal{H}}$ is the set of all PN stuctures for which $\Delta(N)$ is right-closed, as characterized in Section III. Although "$N \in \widehat{\mathcal{H}}$?" is not decidable (Theorem 3), "$N \in \mathcal{K}$?" is decidable by Algorithm 1.

Theorem 5. $\Delta(N)$ *is right-closed if* $N \in \mathcal{K}$.

Proof. Step 2 in Algorithm 1 tests if $\Delta(N^*)$ is control invariant with respect to the uncontrollable transitions in $\mathscr{T}$. If it is control invariant, then by Observation 7, $\Delta(N^*) \subseteq \Delta(N)$. We know $\Delta(N) \subseteq \Delta(N^*)$ and $\Delta(N^*)$ is right-closed. Therefore, $\Delta(N)$ is right-closed if $N \in \mathcal{K}$. □

Fig. 4a shows a PN $N_1 \notin \mathcal{H}$, because t_2 violates the Equation 4. Thus, we have $\mathscr{T} = \{t_2\}$. The PN N^* is shown in Fig. 4b. Here, $\min(\Delta(N_2^*)) = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 1\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$. Consider the element $(1\ 0\ 0\ 0\ 0)^T$. $\max\{(1\ 0\ 0\ 0\ 0)^T, \mathbf{IN}_2\} + \mathbf{C} \times \mathbf{1}_2 = (0\ 1\ 0\ 0\ 0)^T \geq \mathbf{m}^2$. Similarly, $\forall \mathbf{m}^i \in \min(\Delta(N_1^*)), \max\{\mathbf{m}^i, \mathbf{IN}_2\} + \mathbf{C} \times \mathbf{1}_2 \in \Delta(N_1^*)$. Thus, $N_1 \in \mathcal{K}$ and $\Delta(N_1) = \Delta(N_1^*)$ is right-closed.



(a) PN $N_3 \notin \mathcal{K}$ (b) PN N_3^*

Fig. 6: PN $N_3 \notin \mathcal{K}$ and N_3^*. We have $\min(\Delta(N_3)) = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$.

The PN N_2, in Fig. 5a, is neither in $\mathcal{H}$-class nor in $\mathcal{K}$-class. We have $\min(\Delta(N_2^*)) = \{(1\ 0\ 0)^T, (0\ 1\ 0)^T, (0\ 0\ 1)^T\}$. For $\mathbf{m}^1$ and $\mathbf{m}^2$, we can see that $\max\{\mathbf{m}^1, \mathbf{IN}_2\} + \mathbf{C} \times \mathbf{1}_2 = (0\ 0\ 0)^T \notin \Delta(N_2^*)$ and $\max\{\mathbf{m}^2, \mathbf{IN}_2\} + \mathbf{C} \times \mathbf{1}_2 = (0\ 0\ 0)^T \notin \Delta(N_2^*)$. Thus, $N_2 \notin \mathcal{K}$. It can be shown that $\Delta(N_2) = \{((\mathbf{m}(p_1) - \mathbf{m}(p_2))_{mod2} = 1) \vee (\mathbf{m}(p_2) \geq \mathbf{m}(p_1)) \vee (\mathbf{m}(p_3) \geq 1)\}$. $\Delta(N_2)$ is not right-closed, since $(0\ 1\ 0)^T \in \Delta(N_2)$ while $(1\ 1\ 0)^T \notin \Delta(N_2)$.

The PN N_3, in Fig. 6a, belongs neither to $\mathcal{H}$- nor to $\mathcal{K}$-class. $\min(\Delta(N_3^*)) = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$. We have, $\max\{(0\ 0\ 0\ 1\ 0)^T, \mathbf{IN}_2\} + \mathbf{C} \times \mathbf{1}_2 = (0\ 0\ 0\ 0\ 0)^T \notin \Delta(N_3^*)$. Thus, $N_3 \notin \mathcal{K}$. In fact, $(0\ 0\ 0\ 1\ 0)^T \xrightarrow{t_4 t_2} (0\ 0\ 0\ 0\ 0)^T$. Since t_4 is uncontrollable and $(0\ 0\ 0\ 0\ 0)^T \notin \Delta(N_3)$, $(0\ 0\ 0\ 1\ 0)^T \notin \Delta(N_3)$. For this example, we have $\Delta(N_3) = \{(\mathbf{m}(p_1) \geq 1) \vee (\mathbf{m}(p_3) \geq 1) \vee (\mathbf{m}(p_5) \geq 1)\}$, which is right-closed, and $\min(\Delta(N_3)) = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$. The set $\mathbf{P}$, introduced in section III, for N_3 consists of five vectors in $\mathcal{N}^5$ in the set $\{(0\ 1\ 0\ 1\ 0)^T, (1\ 0\ 0\ 0\ 0)^T, (0\ 0\ 1\ 0\ 0)^T, (0\ 0\ 0\ 1\ 0)^T, (0\ 0\ 0\ 0\ 1)^T\}$, corresponding to all uncontrollable transitions respectively. It is not hard to verify that $(\mathbf{m} \in \Delta(N_3)) \Rightarrow ((\mathbf{m} + \mathbf{P}) \in \Delta(N_3))$, which in turn implies that $N_3 \in \widehat{\mathcal{H}}$.

These examples illustrate that $N \in \mathcal{K}$ is only a sufficient condition for right-closure of $\Delta(N)$. Indeed, we proved in Theorem 3 that determining if $\Delta(N)$ is right-closed for an arbitrary PN N is not decidable.

Till now, we showed that restricting the properties of $\Delta(N)$ does not result in decidable instances of the problem of existence of an LESP for $N(\mathbf{m}^0)$. In the next section, we focus on a restricted class of LESPs, i.e. *marking-monotone LESP* (MM-LESP), whose existence for arbitrary PNs is proved to be decidable.

VIII. Marking-Monotone LESP for Arbitrary PNs

An LESP $\mathcal{P}$ for $N(\mathbf{m}^0)$ is an MM-LESP if (1) $\forall \widehat{\mathbf{m}} \geq \mathbf{m}, \forall t \in T, \mathcal{P}(\widehat{\mathbf{m}}, t) \geq \mathcal{P}(\mathbf{m}, t)$, and (2) $\mathcal{P}$ is also an LESP for $N(\widehat{\mathbf{m}}^0)$ for any $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$. For a PN structure N, the set $\Delta_M(N) := \{\mathbf{m}^0 \in \mathcal{N}^n : \exists$ an MM-LESP for $N(\mathbf{m}^0)\}$ is a right-closed subset of $\Delta(N)$. As an example, for the PN structure N_1 shown in Fig. 1, $\Delta(N_1) = \{\mathbf{m} \in \mathcal{N}^5 : (\mathbf{m}(p_1) + \mathbf{m}(p_2) + \mathbf{m}(p_3) + \mathbf{m}(p_4) + \mathbf{m}(p_5) \geq 1)\}$, and $\Delta_M(N_1) = \Delta(N_1)$. The trivial policy of enabling all transitions is an MM-LESP for $N(\mathbf{m}^0)$ for any $\mathbf{m}^0 \in \Delta(N)$. There are some known classes of PNs for which $\Delta_M(N) = \Delta(N)$, and the existence of the minimally restrictive LESP, which is also an MM-policy, for $N(\mathbf{m}^0)$ is decidable [5]–[8].

We are interested in determining whether there is an MM-LESP for an arbitrary PN $N(\mathbf{m}^0)$ (i.e. "*Is* $\mathbf{m}^0 \in \Delta_M(N)$*?*" and "*Is* $\mathbf{m}^0 \notin \Delta_M(N)$*?*"). We present a (decidable) necessary and sufficient condition for the existence of an MM-LESP for $N(\mathbf{m}^0)$. This result involves the *coverability graph* $G(N(\mathbf{m}^0), \mathcal{P}) = G(V, A, \Psi)$, which is essentially the *Karp-Miller tree*, where the duplicate nodes are merged as one (cf. Fig. 1, [5]). More formally, the *coverability graph* of a PN $N(\mathbf{m}^0)$ under the supervision of a marking monotone policy $\mathcal{P}$ is a directed graph $G(N(\mathbf{m}^0), \mathcal{P}) = (V, A, \Psi)$, where V is the set of *vertices*, A is the set of *directed edges*, and $\Psi : A \to V \times V$ is the *incidence function*. For each $a \in A$, if $\Psi(a) = (v_i, v_j)$, then the directed edge a is said to *originate* (*terminate*) at v_i (v_j) (cf. Fig. 1 and 2, [5]). Since each vertex in the coverability graph has at most one outgoing edge labeled by each transition in T, directed paths in the coverability graph can be unambiguously identified by strings in T^*. If there is a path from $v_i \in V$ to $v_j \in V$ with label $\sigma^* \in T^*$ in $G(N(\mathbf{m}^0), \mathcal{P})$, we denote it as $v_i \xrightarrow{\sigma^*} v_j$. Following Reference [5], we say $G(N(\mathbf{m}^0), \mathcal{P})$ satisfies the *path-requirement* if $\exists v_1, v_2 \in V, \exists \sigma_1, \sigma_2 \in T^*$, such that (1) $v_1 \xrightarrow{\sigma_1} v_2 \xrightarrow{\sigma_2} v_2$, (2) $\mathbf{x}(\sigma_2) \geq \mathbf{1}$, that is, all transitions in T appear at least once in σ_2, and (3) $\mathbf{Cx}(\sigma_2) \geq \mathbf{0}$, where $\mathbf{x}(\bullet) \in \mathcal{N}^m$ is an m-dimensional vector, which represents the number of occurrences of each $t \in T$ in the string argument.

Theorem 6. *There is an MM-LESP for* $N(\mathbf{m}^0)$ *if and only if* $\exists \widehat{\Delta}(N) \subseteq \Delta(N)$*, such that*

1) $\mathbf{m}^0 \in \widehat{\Delta}(N)$,
2) $\widehat{\Delta}(N)$ *is control invariant with respect to N,*
3) $\widehat{\Delta}(N)$ *is right-closed, and*
4) $\forall \mathbf{m}^i \in \min(\widehat{\Delta}(N))$, $G(N(\mathbf{m}^i), \mathcal{P})$ *satisfies the path requirement. That is,* $\forall \mathbf{m}^i \in \min(\widehat{\Delta}(N))$*, there is a path* $v_0 \xrightarrow{\sigma_1} v_1 \xrightarrow{\sigma_2} v_1$*, in the coverability graph* $G(N(\mathbf{m}^i), \mathcal{P}) = (V, A)$*, such that* $\mathbf{x}(\sigma_2) \geq \mathbf{1}$ *and* $\mathbf{Cx}(\sigma_2) \geq \mathbf{0}$*, where* $\mathbf{1}$ *is the m-dimensional vector of all ones,* $\mathcal{P}$ *ensures the reachable markings never leaves* $\widehat{\Delta}(N)$.

Proof. (*Only If*) Let $\widehat{\Delta}(N) = \Delta_M(N)$. Since there is an MM-LESP for $N(\mathbf{m}^0)$, it follows that $\mathbf{m}^0 \in \widehat{\Delta}(N)$. By definition, $\widehat{\Delta}(N)$ $(= \Delta_M(N))$ is right-closed. Suppose $\mathbf{m}^1 \in \widehat{\Delta}(N)$ and $\mathbf{m}^1 \xrightarrow{t_u} \mathbf{m}^2$ for some $t_u \in T_u$, then it must be that $\mathbf{m}^2 \in \widehat{\Delta}(N)$ $(= \Delta_M(N))$, as well. Otherwise, the supervisory policy will not be an MM-LESP for $\mathbf{m}^2$. Therefore, $\widehat{\Delta}(N)$ is control invariant with respect to N. In fact, using the same argument, it must be true that the supervisory policy disables any controllable transitions that result in a marking that is not in the set $\widehat{\Delta}(N)$. The supervisory policy $\mathcal{P}$ that ensures the reachable marking never leaves $\widehat{\Delta}(N) = \Delta_M(N)$ is an MM-LESP for $N(\mathbf{m}^0)$. From lemma 5.13, [5], we note that the path-requirement of Theorem 6 is satisfied, as well.

(*If*) Suppose there is a right-closed, control invariant subset $\widehat{\Delta}(N) \subseteq \Delta(N)$, such that $\mathbf{m}^0 \in \widehat{\Delta}(N)$ and each minimal element in $\min(\widehat{\Delta}(N))$ satisfies the path requirement addressed in Theorem 6. We consider a supervisory policy $\mathcal{P}$ for $N(\mathbf{m}^0)$, which prevents any controllable transition whose firing will take the marking outside $\widehat{\Delta}(N)$, and show that $\mathcal{P}$ is a marking-monotone LESP.

Suppose there exists an $\mathbf{m} \in \Re(N, \mathbf{m}^0, \mathcal{P})$ and $t \in T$ such that $\mathcal{P}(\mathbf{m}, t) = 1$. Since this supervisory policy prevents a controllable transition if and only if its firing takes the marking outside $\widehat{\Delta}(N)$, it implies that $\exists \mathbf{m}^j \in \min(\widehat{\Delta}(N))$ such that $\max\{\mathbf{m}, \mathbf{IN}_t\} + \mathbf{C} \times \mathbf{1}_t \geq \mathbf{m}^j$. Now consider all markings larger than $\mathbf{m}$. For all $\widehat{\mathbf{m}} \geq \mathbf{m}, \max\{\widehat{\mathbf{m}}, \mathbf{IN}_t\} + \mathbf{C} \times \mathbf{1}_t \geq \max\{\mathbf{m}, \mathbf{IN}_t\} + \mathbf{C} \times \mathbf{1}_t \geq \mathbf{m}^j$. Since the markings are in $\widehat{\Delta}(N)$ after the firing of t from all $\widehat{\mathbf{m}} \geq \mathbf{m}$, we have $\mathcal{P}(\widehat{\mathbf{m}}, t) = 1$. Hence the supervisory policy is marking-monotone.

Since $\widehat{\Delta}(N)$ is control-invariant with respect to N, and its minimal elements satisfy the path-requirement, there exists an LESP for all markings in $\widehat{\Delta}(N)$ (Theorem 5.1 in [3]). On the other hand, the right-closure property of $\widehat{\Delta}(N)$ indicates that if $\mathbf{m}^0 \in \widehat{\Delta}(N)$ then $\forall \widehat{\mathbf{m}}^0 \geq \mathbf{m}^0, \widehat{\mathbf{m}}^0 \in \widehat{\Delta}(N)$, as well. Thus, $\mathcal{P}$ enforces liveness for $N(\widehat{\mathbf{m}}^0)$, for any $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$. Therefore, the policy is a marking-monotone LESP. □

Algorithm 4, which strongly parallels the procedure in Fig. 8 of reference [5], is a procedure for determining the largest set $\widehat{\Delta}(N)$ that satisfies the properties listed in Theorem 6. Let $\Delta_f(N) \supseteq \Delta(N)$ denote the set of all initial markings for which an LESP exists for N when all transitions are assumed to be controllable. Reference [3] proved that $\Delta_f(N)$ is right-closed and is computable. $\Delta_f(N)$ is the initial estimate of $\Delta(N)$. Algorithm 4 finds, if it exists, by brute force, the largest right-closed control invariant subset of $\Delta_f(N)$, whose minimal elements satisfy the path-requirement. The current estimate of $\Delta(N)$ at any point in the algorithm is denoted by $\widehat{\Upsilon}$. If any of the two properties— control invariance or the path requirement on the coverability graph, is violated then the minimal element that violated the condition is replaced by the smallest set of elements larger than that element, and $\widehat{\Upsilon}$ is appropriately modified. This process is repeated till we find $\Delta_M(N)$ or till $\mathbf{m}^0$ drops out of $\widehat{\Upsilon}$. Algorithms 2 and 3 respectively present procedures for "bumping-up" the minimal elements when the control invariance and path requirement are violated. They take the PN structure N and the current estimate $\widehat{\Upsilon}$ as inputs and respectively output the largest subset of $\widehat{\Upsilon}$ that satisfies the control invariance and path requirement. In Algorithm 4, the PN structure N is the input and the subset $\widehat{\Delta}(N)$ for $N(\mathbf{m}^0)$ is the output.

Algorithm 2 BumpUpForControlInvariance($N, \widehat{\Upsilon}$)

1: **while** $\exists t_u \in T_u$, $\exists \widetilde{\mathbf{m}}^i \in \min(\widehat{\Upsilon})$ such that $(\max\{\mathbf{IN}_{t_u}, \widetilde{\mathbf{m}}^i\} + \mathbf{C} \times \mathbf{1}_{t_u}) \notin \widehat{\Upsilon}$ **do**
2: Replace $\widetilde{\mathbf{m}}^i$ by a set of $k-1$ vectors $\{\widehat{\mathbf{m}}^l\}_{l=1}^{k-1}$ where for each $j \in \{1, 2, \ldots, k\} - \{i\}$, create a new marking $\widehat{\mathbf{m}}^l$, given by the expression $\widehat{\mathbf{m}}^l = \widetilde{\mathbf{m}}^i + max\{\mathbf{0}, \widetilde{\mathbf{m}}^j - (max\{\mathbf{IN} \times \mathbf{1}_{t_u}, \widetilde{\mathbf{m}}^i\} + \mathbf{C} \times \mathbf{1}_{t_u})\}$.
3: Replace the resulting set of $\{\widetilde{\mathbf{m}}^i\}_i$ vectors by their minimal elements, and modify the value of k to equal the size of the minimal set of vectors. $\widehat{\Upsilon}$ is the right-closed set identified by this minimal set of vectors.
4: **end while**

Algorithm 2 aims to compute the supremal controllable subset of the right closed set $\widehat{\Upsilon}$ with respect to the PN structure N. This supremal controllable subset is also right closed.

Algorithm 3 BumpUpForPathRequirement($N, \widehat{\Upsilon}$)

1: **for** $\widetilde{\mathbf{m}}^i \in \min(\widehat{\Upsilon})$ where $G(N(\widetilde{\mathbf{m}}^i), \widetilde{\mathcal{P}}_{\widehat{\Upsilon}})$ does not have the path **do**
2: Define a right-closed set $\overline{\Upsilon}$, where $\min(\overline{\Upsilon}) = (\min(\widehat{\Upsilon}) - \{\widetilde{\mathbf{m}}^i\}) \cup \{\widetilde{\mathbf{m}}^i + \omega \times \mathbf{1}_j | j \in \{1, 2, \ldots, n\}\}$, where $\mathbf{1}_j$ the unit-vector where the j-th component is unity.
3: Replace $\widetilde{\mathbf{m}}^i$ by the set:

$$\{\widetilde{\mathbf{m}}^i + \mathbf{1}_j | j \in \{1, 2, \ldots, n\}, G(N(\widetilde{\mathbf{m}}^i + \omega \times \mathbf{1}_j), \widetilde{\mathcal{P}}_{\overline{\Upsilon}}) \text{ satisfies the path requirement}\} \quad (5)$$

4: **end for**
5: Replace the resulting set of $\{\widetilde{\mathbf{m}}^i\}_i$ vectors by their minimal elements, and modify the value of k to equal the size of the minimal set of vectors. $\widehat{\Upsilon}$ is the right-closed set identified by this minimal set of vectors.

Consequently, when there is an element $\widetilde{\mathbf{m}}^i$ that violates the control invariance requirement, it is elevated by an appropriate minimal amount, as stated in Step 2. During this elevation process, it might happen that we get some minimal elements that are ordered (that is, $\widetilde{\mathbf{m}}^i \geq \widetilde{\mathbf{m}}^j$ for some i, j). Step 3 trims the set of minimal elements of the current version of $\widehat{\Upsilon}$ to ensure that only the smallest elements are retained. This process proceeds until (the current version of) $\widehat{\Upsilon}$ is control invariant.

Proceeding under the stipulation that (the current version of) $\widehat{\Upsilon}$ is control invariant with respect to N, for any $\mathbf{m}^0 \in \widehat{\Upsilon}$, there is a supervisory policy, $\widetilde{P}_{\widehat{\Upsilon}}$, that ensures $\Re(N, \mathbf{m}^0, \widetilde{P}_{\widehat{\Upsilon}}) \subseteq \widehat{\Upsilon}$. If $\forall \widetilde{\mathbf{m}}^i \in min(\widehat{\Upsilon})$, the required path condition in $G(N(\widetilde{\mathbf{m}}^i), \widetilde{P}_{\widehat{\Upsilon}})$ is satisfied, then $\widetilde{P}_{\widehat{\Upsilon}}$ is an LESP for $N(\mathbf{m}^0)$ for any $\mathbf{m}^0 \in \widehat{\Upsilon}$ (cf. [5]).

In Algorithm 3, when there exists an element $\widetilde{\mathbf{m}}^i \in \mathcal{N}^n$ where $G(N(\widetilde{\mathbf{m}}^i), \widetilde{P}_{\widehat{\Upsilon}})$ does not have the required path, it should be elevated by an appropriate set of unit-vectors. Step 2 identifies those among the n-many unit-vectors that are to be used to elevate the minimal element $\widetilde{\mathbf{m}}^i$. Specifically, if the placement of an unbounded number of tokens (i.e. ω-many tokens) in just the j-th place does *not* result in a coverability graph with the required path, then the j-th unit vector is *not* used to elevate the minimal element $\widetilde{\mathbf{m}}^i$. Otherwise, the corresponding vector $\widetilde{\mathbf{m}}^i + \mathbf{1}_j$ is retained in the current set $\overline{\Upsilon}$, as shown in Step 3. This process proceeds until all elements satisfy the path requirement.

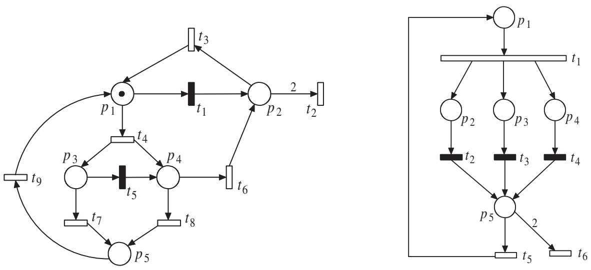
Theorem 7. *The existence of an MM-LESP for an arbitrary PN $N(\mathbf{m}^0)$ is decidable.*

Proof. The existence (non-existence) of an MM-LESP for $N(\mathbf{m}^0)$ is subject to the existence (resp. non-existence) of a proper subset $\widehat{\Delta}(N)$ which is right-closed, control invariant, satisfies the path-requirement and contains $\mathbf{m}^0$. In Algorithm 4, we seek a sequence of proper subsets $\widehat{\Upsilon}$ (where $\widehat{\Delta}(N) \subseteq \widehat{\Upsilon} \subseteq \Delta(N) \subseteq \Delta_f(N)$) using exhaustive search until we find such a $\widehat{\Delta}(N)$ or until $\mathbf{m}^0 \notin \widehat{\Upsilon}$.

If there is an MM-LESP in $N(\mathbf{m}^0)$, from Lemma 5.13 in [5] we know that there is a finite set of minimal elements

Algorithm 4 Test for existence of the subset $\widehat{\Delta}(N)$ for $N(\mathbf{m}^0)$.

$\widehat{\Upsilon} = \Delta_f(N)$, and let $\{\widetilde{\mathbf{m}}^i\}_{i=1}^k = \min(\widehat{\Upsilon})$.
while $((\exists t_u \in T_u, \exists \widetilde{\mathbf{m}}^i \in \min(\widehat{\Upsilon})$, such that $\max\{\widetilde{\mathbf{m}}^i, \mathbf{IN}_u\} + \mathbf{C}\cdot\mathbf{1}_u \notin \widehat{\Upsilon}) \vee (\exists \widetilde{\mathbf{m}}^i \in \min(\widehat{\Upsilon})$ such that $G(N(\widetilde{\mathbf{m}}^i), \widetilde{\mathcal{P}}_{\widehat{\Upsilon}})$ does not have the path requirement$)) \wedge (\mathbf{m}^0 \in \widehat{\Upsilon})$ **do**
  BumpForControlInvariance$(N, \widehat{\Upsilon})$
  BumpForPathCondition$(N, \widehat{\Upsilon})$
end while
if $\mathbf{m}^0 \notin \widehat{\Upsilon}$ **then**
  return(“no solution”);
else
  return $\widehat{\Upsilon}$
end if



(a) PN N_4 (b) PN N_5

Fig. 7: Examples to illustrate features of marking-monotone LESP.

$\{\widetilde{\mathbf{m}}^i\}_{i=1}^k$ that define a control invariant, right-closed set and satisfy the path-requirement. The k-many, n-dimensional, minimal elements $\{\widetilde{\mathbf{m}}^i\}_{i=1}^k$ are determined using brute force by Algorithm 4, in finite time. On the other hand, this process will terminate when $\mathbf{m}^0 \notin \widehat{\Upsilon}$, thus certifying the non-existence of a candidate $\Upsilon = \widehat{\Delta}(N)$ with $\mathbf{m}^0 \in \widehat{\Delta}(N)$, in finite time; thus proving the semi-decidability of the existence (non-existence) of a MM-policy that enforces liveness in an arbitrary partially controllable PN $N(\mathbf{m}^0)$. □

Figure 7a presents an example $N_4(\mathbf{m}^0)$ where $\Delta(N_4)$ is not right-closed, but there is an MM-LESP for $N(\mathbf{m}^0)$. Specifically, $\Delta(N_4) = \{\mathbf{m}^0 \in \mathcal{N}^5 \mid (\mathbf{m}^0(p_1) + \mathbf{m}^0(p_3) + \mathbf{m}^0(p_5) \geq 1) \vee ((\mathbf{m}^0(p_2) + \mathbf{m}^0(p_4))_{mod2} = 1)\}$. Here $\Delta_M(N_4) = \{\mathbf{m}^0 \in \mathcal{N}^5 \mid \mathbf{m}^0(p_1) + \mathbf{m}^0(p_3) + \mathbf{m}^0(p_5) \geq 1\}$. The MM-LESP, $\mathcal{P}$, will ensure at least one token in $\{p_1, p_3, p_5\}$. However, it is to be noted that $\mathcal{P}$ is not the minimally restrictive for $N_4(\mathbf{m}^0)$, since $\Delta_M(N_4) \subset \Delta(N_5)$.

On the other hand, the existence of a right-closed subset of $\Delta(N)$ does not guarantee the existence of a MM-LESP. Consider N_5 with initial marking $(0\ 1\ 0\ 0\ 1)^T$. Following the procedures explicated in Algorithm 4, we end up with the control invariant $\widehat{\Upsilon}$ represented by $\min(\widehat{\Upsilon}) = \{\widetilde{\mathbf{m}}^i\}_{i=1}^7 = \{(1\ 0\ 0\ 0\ 0)^T, (0\ 2\ 0\ 0\ 0)^T, (0\ 1\ 1\ 0\ 0)^T, (0\ 1\ 0\ 1\ 0)^T, (0\ 0\ 2\ 0\ 0)^T, (0\ 0\ 1\ 1\ 0)^T, (0\ 0\ 0\ 2\ 0)^T\}$. Since the initial marking $\mathbf{m}^0$ is not in $\widehat{\Upsilon}$, there is no MM LESP for $N_5(\mathbf{m}^0)$. However, there is an LESP for $N_5(\mathbf{m}^0)$. Note that $\Delta(N_5) = \{\mathbf{m}^0 \in \mathcal{N}^5 \mid \mathbf{m}^0(p_1) + \mathbf{m}^0(p_2) + \mathbf{m}^0(p_3) + \mathbf{m}^0(p_4) \geq 1$ or $\mathbf{m}^0(p_5)_{mod2} = 1\}$. A supervisory policy that ensures that there is at least one token in $\{p_1, p_2, p_3, p_4\}$ or there are odd tokens in p_5 is an LESP. This example also shows that even if there is no marking-monotone LESP for arbitrary PN $N(\mathbf{m}^0)$, there may be an LESP for $N(\mathbf{m}^0)$.

IX. Conclusion

This paper is about the existence of a *liveness enforcing supervisory policy* (LESP) for an arbitrary *Petri net* (PN) $N(\mathbf{m}^0)$, where $N = (\Pi, T, \Phi, \Gamma)$, and $\mathbf{m}^0 : \Pi \rightarrow \mathcal{N}$ is the *initial-marking*. The set

$$\Delta(N) := \{\mathbf{m}^0 \mid \exists \text{ an LESP for } N(\mathbf{m}^0)\},$$

is *control invariant* with respect to N and plays a critical role in deciding if there is an LESP for the PN $N(\mathbf{m}^0)$. Specifically, there is an LESP for $N(\mathbf{m}^0)$ if and only if $\mathbf{m}^0 \in \Delta(N)$.

In prior work, we proved that neither the membership nor the non-membership of a marking in $\Delta(N)$ is semi-decidable for an arbitrary PN structure. In this paper, we generalized this decision problem and showed that neither “*Is* $\Delta(N) = \emptyset$*?*” nor “*Is* $\Delta(N) \neq \emptyset$*?*” is semi-decidable.

An integer-valued set of vectors is said to be *right-closed* if the presence of a vector in the set implies that all term-wise larger vectors are also in the set. We presented a necessary and sufficient condition for $\Delta(N)$ to be right-closed for an arbitrary PN. Following this, we showed that “*Is* $\Delta(N)$ *right-closed?*” is undecidable for arbitrary PN structures. We also showed that for arbitrary PN structures the decision problems: “*Is there a right-closed subset of* $\Delta(N)$*?*” and “*Is there no right-closed subset of* $\Delta(N)$*?*” are not semi-decidable.

If a transition is control-enabled at some marking under the supervision of a *marking-monotone policy* (MM-policy), then it is control-enabled at all larger markings as well. An MM-policy $\mathcal{P}$ is a *marking-monotone LESP* (MM-LESP) for $N(\mathbf{m}^0)$ if it is an LESP for $N(\widehat{\mathbf{m}}^0)$ for all $\widehat{\mathbf{m}}^0 \geq \mathbf{m}^0$, as well. The set

$$\Delta_M(N) := \{\mathbf{m}^0 \mid \exists \text{ an MM-LESP for } N(\mathbf{m}^0)\},$$

is a right-closed subset of $\Delta(N)$ for any PN structure N. After introducing a class of PN structures for which the set $\Delta(N)$ is known to be right-closed, we showed that the existence of an MM-LESP for an arbitrary PN $N(\mathbf{m}^0)$ is decidable. That is, “*Is* $\mathbf{m}^0 \in \Delta_M(N)$*?*” is decidable for any PN structure N. Thus, starting from the two decision problems: *“Is* $\mathbf{m}^0 \in \Delta(N)$?″ and *“Is* $\mathbf{m}^0 \notin \Delta(N)$?″ that are not semi-decidable, we present a string of results that culminates in decidable sub-problems: *“Is* $\mathbf{m}^0 \in \Delta_M(N)$?″ and *“Is* $\mathbf{m}^0 \notin \Delta_M(N)$?″.

These results lead to the conclusion that extracting any kind of information about $\Delta(N)$ for an arbitrary PN is most likely an extremely hard problem. Besides, we can also conclude that between the properties of the set of initial markings for which an LESP exists, and the characteristics of the LESP, it is the characteristics of the LESP that plays a prominent role in determining decidability. That is, if a supervisory policy $\mathcal{P}$ is such that $\Re(N, \mathbf{m}, \mathcal{P})$ (which can have an unbounded number of markings) can be reduced to a reachability graph with a finite number of appropriately defined symbolic markings such that the liveness property is preserved, then the existence of $\mathcal{P}$ is likely to be decidable. Marking Monotone LESPs is one instance of such an LESP in which a reachability graph with possibly infinite number of markings is reduced to a coverability graph with finite number of nodes while preserving liveness. The idea is as follows. Recall that for an MM-LESP $((\mathcal{P}(\mathbf{m}, t) = 1) \Rightarrow (\mathcal{P}(\widehat{\mathbf{m}}, t) = 1))\ \forall \widehat{\mathbf{m}} \geq \mathbf{m}$. Intuitively, from the perspective of the supervisory policy, every marking larger than $\mathbf{m}$ is the same as $\mathbf{m}$ (as the supervisory action is the same). Therefore, all markings larger than a minimal element (at which the supervisory policy permits a transition) in the reachability graph can be replaced by the minimal element itself; thereby reducing an unbounded number of markings to a single marking. That said, it might not be the only such class of LESPs. The results presented in the paper open up new avenues of research and provides a guideline for future research aimed at identifying classes of PNs for which the existence of an LESP is decidable.

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Chen Chen received the B.S. degree in Automation from Zhoukou Normal University, Zhoukou, Henan, China, in 2012, and Ph. D degree in Electromechanical Engineering from Xidian University, Xi'an, Shaanxi, China, in 2019. She is currently a Lecturer at Xidian University. Her research interests are discrete event systems and their supervisory control techniques, Petri nets, automated manufacturing systems, and supervisor simplification.

Arun Raman is currently working toward the Ph.D degree in Systems Engineering at University of Illinois at Urbana-Champaign, Urbana, IL, USA. He received his Masters in Systems and Control Engineering from Indian Institute of Technology Bombay, Mumbai, India following which he worked in GE Oil and Gas, in Compressor and Gas Turbine simulation, modeling and control. His research interests lie in the area of classical control, applied mathematics and control of discrete-event systems.

HeSuan Hu (M'11-SM'12) received the B.S. degree in computer engineering and the M.S. and Ph. D degrees in Electro-mechanical Engineering from Xidian University, Xi'an, Shaanxi, China, in 2003, 2005, and 2010, respectively. He is currently a Full Professor at Xidian University. His research interests include discrete event systems and their supervisory control techniques, Petri nets, automated manufacturing systems, multimedia streaming systems, and artificial intelligence. In the aforementioned areas, he has over 120 publications in journals, book chapters, and conference proceedings. Dr. Hu is an Associate Editor for IEEE Control Systems Magazine, IEEE Robotics & Automation Magazine, IEEE Transactions on Automation Science and Engineering, Journal of Intelligent Manufacturing, etc. He was on the Editorial Board of 10+ international journals. He was the recipient of many national and international awards including the Franklin V. Taylor Outstanding Paper Award from the IEEE SMC Society in 2010 and the finalists of the Best Automation Paper from the IEEE RAS Society in 2013, 2016, and 2017.

R. S. Sreenivas (S'83-M'93-SM'02) received the B.Tech. degree in electrical engineering from the Indian Institute of Technology, Madras, India, in 1985, and the M.S. and Ph.D. degrees in electrical and computer engineering from Carnegie Mellon University, Pittsburgh, PA in 1987 and 1990, respectively. He was a Postdoctoral Fellow in Decision and Control at the Division of Applied Sciences, Harvard University, Cambridge, MA, before he joined the University of Illinois at Urbana-Champaign in September 1992. He is currently a Professor of Industrial and Enterprise Systems Engineering, Research Professor with the Coordinated Science Laboratory, Information and Trust Institute, and the Center for Autonomy at the University of Illinois. His research interests include modeling, analysis, control and performance evaluation of discrete-event/discrete-state systems.