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Brief Paper


# An Application of Independent, Increasing, Free-Choice Petri Nets to the Synthesis of Policies that Enforce Liveness in Arbitrary Petri Nets*

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**Key Words**—Discrete event dynamic systems; supervisory control; petri nets.


**Abstract**—The class of *Independent, Increasing, Free-Choice Petri nets* (II-FCPNs) was introduced in (Sreenivas, 1997c), where it is shown that any II-FCPN can be made *live* via supervision using a readily available policy. In a *live Petri net* (PN) (Peterson (1981). Petri Net Theory and Modeling of systems. Prentice-Hall, Englewood Cliffs, NJ, Reisig (1985). Petri Nets. Springer, Berlin), it is possible to fire any transition from every reachable marking, although not necessarily immediately. In this paper we identify a class of PNs, where every transition is controllable, that are not necessarily II-FCPNs, that can be made live via supervision using a readily available policy constructed from the policy that enforces liveness in an II-FCPN. © 1998 Elsevier Science Ltd. All rights reserved.


## 1. *Introduction*

We consider *Petri net* (PN) (Peterson, 1981; Reisig, 1985; Murata, 1989) models of *discrete event dynamic systems* (DEDS), where every transition is controllable, and we concern ourselves with the property of *liveness* in the PN models of DEDS. A PN is said to be *live* if it is possible to fire any transition from every reachable marking, although not necessarily immediately. In general, investigating the existence, and the synthesis of such policies can be computationally expensive (Sreenivas, 1997d). However, in this paper we show that if the *free-choice equivalent* (Sreenivas, 1997b) of the original PN belongs to a restricted class called *independent, increasing, free-choice petri nets* (II-FCPNs) (Sreenivas, 1997c), there is a readily available supervisory policy that enforces liveness in the original PN.

Section 2 contains the notational preliminaries. Section (2) presents the main results of this paper, and in Section 4 we conclude with suggested future research topics.

## 2. *Notations and definitions*

A *Petri net* (PN) $N = (\Pi, T, \Phi, \mathbf{m}^0)$ is an ordered 4-tuple, where $\Pi = \{p_1, p_2, \ldots, p_n\}$ is a set of *n places*, $T = \{t_1, t_2, \ldots, t_m\}$ is a set of *m transitions*, $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*, $\mathbf{m}^0 : \Pi \to \mathcal{N}$ is the *initial marking function* (or the *initial marking*), and $\mathcal{N}$ is the set of nonnegative integers. The *state* of a PN is the marking $\mathbf{m} : \Pi \to \mathcal{N}$ that identifies the number of *tokens* in each place. A PN $N$ is *strongly connected* if the reflexive, transitive closure of $\Phi$ equals the set $(\Pi \cup T) \times (T \cup \Pi)$. For a given marking $\mathbf{m}$ a transition $t \in T$ is said to be *enabled* if $\forall p \in {}^{\bullet}t$, $\mathbf{m}(p) \geq 1$, where ${}^{\bullet}x := \{y | (y, x) \in \Phi\}$. The set of enabled transitions is denoted by the symbol $T_e(\mathbf{m})$. An enabled transition $t \in T_e(\mathbf{m})$ can *fire*, which changes the marking $\mathbf{m}$ to $\hat{\mathbf{m}}$ according to the equation

$$\hat{\mathbf{m}}(p) = \mathbf{m}(p) - \mathrm{card}(p^{\bullet} \cap \{t\}) + \mathrm{card}({}^{\bullet}p \cap \{t\}), \qquad (1)$$

where $x^{\bullet} := \{y | (x, y) \in \Phi\}$, and the symbol card($\bullet$) is used to denote the cardinality of the set argument. This notation is also used to denote the predecessor or successor set of a set of places or transitions. A collection of places $P \subseteq \Pi$ is said to be a *siphon* (*trap*) if ${}^{\bullet}P \subseteq P^{\bullet}$ ($P^{\bullet} \subseteq {}^{\bullet}P$). A trap (siphon) $P$, is said to be *minimal* if $\nexists \tilde{P} \subset P$, such that $\tilde{P}^{\bullet} \subseteq {}^{\bullet}\tilde{P}$ (${}^{\bullet}\tilde{P} \subseteq \tilde{P}^{\bullet}$).

A string of transitions $\sigma$ is said to be a *valid firing string* starting from the marking $\mathbf{m}$, if after the firing of each transition in the sequence denoted by $\sigma$ the marking remains non-negative. The set of *reachable markings* from $\mathbf{m}^0$, denoted by $\Re(N, \mathbf{m}^0)$, is the set of markings generated by all valid firing strings starting with marking $\mathbf{m}^0$ in the PN $N$. At a marking $\mathbf{m}^1$, if the firing of a valid firing string $\sigma$ results in a marking $\mathbf{m}^2$, we represent it as $\mathbf{m}^1 \to \sigma \to \mathbf{m}^2$. A transition $t \in T$ is *live* if $\forall \mathbf{m}^1 \in \Re(N, \mathbf{m}^0)$, $\exists$ a $\mathbf{m}^2 \in \Re(N, \mathbf{m}^1)$ such that $t \in T_e(\mathbf{m}^2)$.

For any $\sigma \in T^*$, and for any $\hat{T} \subseteq T$, the *projection* of $\sigma$ onto the subset $\hat{T}$, denoted by $\sigma|_{\hat{T}}$ is the string of transitions in $\hat{T}$ obtained from $\sigma$ by erasing all transitions that are not in $\hat{T}$ while retaining the transition order in the rest of the string.

A *supervisory policy* $\mathscr{P} : \mathcal{N}^n \to \{0, 1\}^m$, is a total map that returns an $m$-dimensional binary vector for each reachable marking. The supervisory policy $\mathscr{P}$ permits the firing of transition $t_i$ at marking $\mathbf{m}$, only if $\mathscr{P}(\mathbf{m})_i \neq 1$. A string of transitions $\sigma = t_{j_1} t_{j_2} \ldots t_{j_k}$ is said to be a *valid firing string under the supervision of* $\mathscr{P}$ starting from the marking $\mathbf{m}$, if (i) the transition $t_{j_1}$ is enabled at the marking $\mathbf{m}$, $\mathscr{P}(\mathbf{m})_{j_1} = 1$, and (ii) for $i \in \{1, 2, \ldots, k-1\}$ the firing of the transition $t_{j_i}$ produces a marking $\hat{\mathbf{m}}$ at which the transition $t_{j_{i+1}}$ is enabled and $\mathscr{P}(\hat{\mathbf{m}})_{j_{i+1}} = 1$. The set of reachable markings under the supervision of $\mathscr{P}$ in $N$ from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathscr{P})$. A transition $t_{j_i}$ is *live* under the supervision of $\mathscr{P}$ if $\forall \mathbf{m} \in \Re(N, \mathbf{m}^0, \mathscr{P})$, $\exists \hat{\mathbf{m}} \in \Re(N, \mathbf{m}, \mathscr{P})$ such that $t_{j_i} \in T_e(\hat{\mathbf{m}})$ and $\mathscr{P}(\hat{\mathbf{m}})_{j_i} = 1$.

A supervisory policy $\mathscr{P}$ enforces liveness if all transitions in $N$ are live under $\mathscr{P}$. Sreenivas (1997d) presents a necessary and sufficient condition for the existence of a supervisory policy that enforces liveness in an arbitrary PN. This condition is untestable in general, but it is testable if the PN is bounded, or, if each transition in the PN can be prevented from firing by the supervisory policy. Testing the existence of a supervisory policy that enforces liveness in an arbitrary, bounded PN is at least PSPACE-complete (Sreenivas, 1997d). The class of PNs we consider in this paper are not necessarily bounded, but we assume each transition can be prevented from firing by the supervisory policy.

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A PN $N = (\Pi, T, \Phi, \mathbf{m}^0)$ is a *free-choice* PN (FCPN) if $\forall p \in \Pi$, $\mathrm{card}(p^\bullet) > 1 \Rightarrow {}^\bullet(p^\bullet) = \{p\}$. The class of policies that enforce liveness in FCPNs is characterized in (Sreenivas (1997c). This reference also introduced a new class of FCPNs called *independent, increasing, free-choice Petri nets* (II-FCPNs), and it is shown that any II-FCPN can be made live via supervision using a readily available supervisory policy. We now present the definition of II-FCPNs.

For an arbitrary FCPN $\tilde{N} = (\tilde{\Pi}, \tilde{T}, \tilde{\Phi}, \tilde{\mathbf{m}}^0)$, an *independent, increasing siphon* (II-siphon) is a siphon $P \subseteq \tilde{\Pi}$ (${}^\bullet P \subseteq P^\bullet$) where $\exists\ \hat{T} \subseteq {}^\bullet P$ such that (i) $\hat{T}^\bullet = {}^\bullet\hat{T} \subseteq P$, (ii) $\exists\ \hat{t} \in \hat{T}$, such that $\mathrm{card}(\hat{t}^\bullet) > 1$, (iii) the FCPN induced by the set of places ${}^\bullet\hat{T}$, and set of transitions $\hat{T}$, is strongly connected, and (iv) there is at least one token in ${}^\bullet\hat{T}$ at the initial marking. The set of transitions $\hat{T}$ is called the *critical transition set* of the II-siphon. The FCPN induced by the set of places ${}^\bullet\hat{T}$, and set of transitions $\hat{T}$ is referred to as the *strongly connected component* of the II-siphon. An *independent, increasing FCPN* (II-FCPN) is an FCPN where every minimal siphon either (i) contains a marked trap at the initial marking, or (ii) is an II-siphon. In Sreenivas (1997c) it is shown that any supervisory policy that maintains at least one token in the input place-set (${}^\bullet\hat{T}$, for example) of each II-siphon enforces liveness.

Any arbitrary PN $N = (\Pi, T, \Phi, \mathbf{m}^0)$ can be converted to an equivalent FCPN $\tilde{N} = (\tilde{\Pi}, \tilde{T}, \tilde{\Phi}, \tilde{\mathbf{m}}^0)$ with the addition of a few extra places and transitions (Hack, 1974). The conversion procedure is outlined in Fig. 1 of Sreenivas (1997b). Additionally, in this reference it is shown that there exists a supervisory policy that enforces liveness in the PN $N$ if and only if there exists a corresponding policy for the FCPN $\tilde{N}$. The PN $N$ shown in Fig. 2a of Sreenivas (1997b) is not an FCPN. To see this note that ${}^\bullet(p_9^\bullet) = {}^\bullet(p_{10}^\bullet) = \{p_9, p_{10}\}$. The PN $\tilde{N}$ shown in Fig. 2b of this reference is an FCPN obtained by using the procedure convert_to_FCPN ($N$). In Section 4.1 of Sreenivas (1997c) it is shown that the FCPN $\tilde{N}$ is an II-FCPN. The repetition of these observations are skipped for brevity.

Since the PN $N$ and the FCPN $\tilde{N}$ share common places and transitions it is necessary to define the following sets to eliminate any confusion. The symbol $({}^\bullet p)_N$ denotes the set $\{t \in T \mid (t, p) \in \Phi\}$, and the symbol $({}^\bullet p)_{\tilde{N}}$ denotes the set $\{t \in \tilde{T} \mid (t, p) \in \tilde{\Phi}\}$. The symbols $(p^\bullet)_N$, $(p^\bullet)_{\tilde{N}}$, $({}^\bullet t)_N$, $({}^\bullet t)_{\tilde{N}}$, $(t^\bullet)_N$, and $(t^\bullet)_{\tilde{N}}$ are defined accordingly.

In the following section, we consider the family of PNs $N = (\Pi, T, \Phi, \mathbf{m}^0)$ that yield an II-FCPN when the procedure *convert_to_FCPN* is applied to $N$. That is, we consider the class of PNs $N$, such that $(\tilde{N}, \tilde{T}, \tilde{\Pi}) =$ *convert_to_FCPN* ($N$), results in $\tilde{N}$ being an II-FCPN. We show that the supervisory policy of maintaining at least one token in the input place-set of each II-siphon in $\tilde{N}$ can be modified to yield a policy that enforces liveness in the PN $N$.

### 3. *Main results*

Let $N = (\Pi, T, \Phi, \mathbf{m}^0)$ be PN, that is not necessarily an II-FCPN, such that if $(\tilde{N}, \tilde{T}, \tilde{\Pi}) =$ *convert_to_FCPN* ($N$), the FCPN $\tilde{N} = (\tilde{\Pi}, \tilde{T}, \tilde{\Phi}, \tilde{\mathbf{m}}^0)$ is an II-FCPN. We show that there exists a readily constructible supervisory policy that enforces liveness in $N$. In Lemmas 1 and 2, we make a few observations on the influence of the supervisory policy $\mathscr{P}$ that maintains at least one token in the input place set of the critical transition set of each II-siphon in the II-FCPN $\tilde{N}$. Lemma 1 establishes the fact that the sum of the token loads of the places in each II-siphon of $\tilde{N}$ cannot increase by firing transitions in $\tilde{T}$, this observation follows directly from the definition of a II-siphon. In Lemma 2 we show that if the supervisory policy $\tilde{\mathscr{P}}$ prevents the firing of some state-enabled transition at an arbitrary marking, then this transition can only have one input place, furthermore, this single input place is a member of the place set $\Pi$, the set of places in the PN $\tilde{N}$ that originally belonged to the PN $N$.

*Lemma* 1. Let $\tilde{\mathscr{P}}: \mathcal{N}^{\mathrm{card}(\tilde{\Pi})} \to \{0, 1\}^{\mathrm{card}(\tilde{\Pi})}$ denote the supervisory policy that maintains at least one token in the input place set of the critical transition set of each II-siphon in the II-FCPN $\tilde{N}$. Let $\tilde{\mathbf{m}}^1 \to \tilde{\sigma} \to \tilde{\mathbf{m}}^2$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$, and $\tilde{\sigma} \in \tilde{T}^*$. Then, for each II-siphon $P \subseteq \tilde{\Pi}$ (${}^\bullet P \subseteq P^\bullet$),

$$\sum_{p \in P} \tilde{\mathbf{m}}^1(p) \geq \sum_{p \in P} \tilde{\mathbf{m}}^2(p).$$

*Lemma* 2. Let $\tilde{\mathscr{P}}: \mathcal{N}^{\mathrm{card}(\tilde{\Pi})} \to \{0, 1\}^{\mathrm{card}(\tilde{\Pi})}$ denote the supervisory policy that maintains at least one token in the input place set of the critical transition set of each II-siphon in the II-FCPN $\tilde{N}$. If $\tilde{\mathscr{P}}$ prevents the firing of some transition $t_i \in \tilde{T}$ at a marking $\tilde{\mathbf{m}}$, where $t_i \in T_e(\tilde{\mathbf{m}})$ (i.e. $\tilde{\mathscr{P}}(\tilde{\mathbf{m}})_i = 0$), then (i) $\mathrm{card}({}^\bullet t_i) = 1$, and (ii) ${}^\bullet t_i \cap \hat{\Pi} = \emptyset$.

*Proof.* If $t_i \in T_e(\tilde{\mathbf{m}})$ and $\tilde{\mathscr{P}}(\tilde{\mathbf{m}})_i = 0$, then there exists an II-siphon $P$ with a critical transition set $\hat{T}$, such that $t_i \notin \hat{T}$ and ${}^\bullet t_i \cap {}^\bullet\hat{T} \neq \emptyset$. Additionally, permitting the firing of $t_i$ at the marking $\tilde{\mathbf{m}}$ will empty the place set ${}^\bullet\hat{T}$. Since ${}^\bullet\hat{T} = \hat{T}^\bullet$ and ${}^\bullet t_i \cap {}^\bullet\hat{T} \neq \emptyset$, we infer $\exists\ t_j \in \hat{T}$ such that ${}^\bullet t_i = {}^\bullet t_j$. Since $\tilde{N}$ is an FCPN, we conclude $\mathrm{card}({}^\bullet t_i) = 1$. If ${}^\bullet t_i \cap \hat{\Pi} \neq \emptyset$, then by construction it follows that $\mathrm{card}({}^\bullet t_i) > 1$. Hence ${}^\bullet t_i \cap \hat{\Pi} = \emptyset$. □

Following Sreenivas (1997b), let $\mathbf{m} \in \mathcal{N}^{\mathrm{card}(\Pi)}$ be a marking of the PN $N$, we define a corresponding marking $\beta(\mathbf{m})$ of the PN $\tilde{N}$ as $\beta(\mathbf{m})\ (p) = \mathbf{m}(p)$ (0) if $p \in \Pi$ (otherwise). Conversely, if $\tilde{\mathbf{m}} \in \mathcal{N}^{\mathrm{card}(\tilde{\Pi})}$ is a marking of the PN $\tilde{N}$, we define a corresponding marking $\alpha(\tilde{\mathbf{m}})$ of the PN $N$ as follows $\alpha(\tilde{\mathbf{m}})\ (p) = \tilde{\mathbf{m}}(p) + \sum_{\hat{p} \in ((p^\bullet)^\bullet_{\tilde{N}}) \cap \hat{\Pi}} \tilde{\mathbf{m}}(\hat{p})$. The following lemma guarantees the existence of a particular firing string in the II-FCPN $\tilde{N}$ under the supervision of the policy $\tilde{\mathscr{P}}$ introduced earlier. The result is established by an induction argument over the length of the string $\tilde{\sigma}_1|_T$. The details of this proof is skipped for brevity and can be found in (Sreenivas, 1997a).

*Lemma* 3. Let $\tilde{\mathscr{P}}: \mathcal{N}^{\mathrm{card}(\tilde{\Pi})} \to \{0, 1\}^{\mathrm{card}(\tilde{\Pi})}$ denote the supervisory policy that maintains at least one token in the input place set of the critical transition set of each II-siphon in the II-FCPN $\tilde{N}$. If $\tilde{\mathbf{m}}^0 \to \tilde{\sigma}_1 \to \tilde{\mathbf{m}}^1$ under the supervision of $\tilde{\mathscr{P}}$, then $\exists\ \hat{\sigma}_1 \in \tilde{T}^*$, such that $\beta(\alpha(\tilde{\mathbf{m}}^0)) \to \hat{\sigma}_1 \to \beta(\alpha(\tilde{\mathbf{m}}^1))$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$, and $\hat{\sigma}_1|_T = \tilde{\sigma}_1|_T$.

In Lemma 4 we present a property of a supervisory policy $\mathscr{P}: \mathcal{N}^{\mathrm{card}(\Pi)} \to \{0, 1\}^{\mathrm{card}(\Pi)}$, to be used on the original PN $N$, that is derived from the supervisory policy $\tilde{\mathscr{P}}$ for the II-FCPN $\tilde{N}$. A transition $t_i \in T$ is control-enabled in $N$ under the supervisory policy $\mathscr{P}$ at a marking $\mathbf{m}$ if and only if $\exists\ \tilde{\sigma} \in \tilde{T}^*$, such that $\beta(\mathbf{m}) \to \tilde{\sigma} \to \tilde{\mathbf{m}}$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$ and $t_i$ is control- and state-enabled in $\tilde{N}$ at the marking $\tilde{\mathbf{m}}$.

*Lemma* 4. Let $\mathscr{P}: \mathcal{N}^{\mathrm{card}(\Pi)} \to \{0, 1\}^{\mathrm{card}(\Pi)}$ be the supervisory policy for the PN $N$ derived from the policy $\tilde{\mathscr{P}}$ as defined above. If $\tilde{\mathbf{m}}^0 \to \tilde{\sigma}_1 \to \tilde{\mathbf{m}}^1$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$, then $\alpha(\tilde{\mathbf{m}}^0) \to \tilde{\sigma}_1|_T \to \alpha(\tilde{\mathbf{m}}^1)$ under the supervision of $\mathscr{P}$ in $N$.

*Proof.* If $\tilde{\mathbf{m}}^0 \to \tilde{\sigma}_1 \to \tilde{\mathbf{m}}^1$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$, then from Lemma 3, we know $\beta(\alpha(\tilde{\mathbf{m}}^0)) \to \tilde{\sigma}_2 \to \beta(\alpha(\tilde{\mathbf{m}}^1))$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$ and $\tilde{\sigma}_1|_T = \tilde{\sigma}_2|_T$. We will show that if $\beta(\alpha(\tilde{\mathbf{m}}^0)) \to \tilde{\sigma}_1 \to \beta(\alpha(\tilde{\mathbf{m}}^1))$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$, then $\alpha(\tilde{\mathbf{m}}^0) \to \tilde{\sigma}_1|_T \to \alpha(\tilde{\mathbf{m}}^1)$ under the supervision of $\mathscr{P}$ in $N$. This observation is established by induction on the length of $\tilde{\sigma}_1|_T$.

The base case is established when $\tilde{\sigma}_1|_T$ is empty-string. As the induction hypothesis assumes the above claim is true for any $\tilde{\sigma}_1|_T$ of length less than or equal to $n$ for some $n \in \mathcal{N}$. For the induction step let $\beta(\alpha(\tilde{\mathbf{m}}^0)) \to \tilde{\sigma}_1 \to \beta(\alpha(\tilde{\mathbf{m}}^1)) \to \tilde{\sigma}_2 \to \beta(\alpha(\tilde{\mathbf{m}}^2))$, where $\tilde{\sigma}_2|_T = t_i$, $t_i \in T$. From the induction hypothesis we infer $\alpha(\tilde{\mathbf{m}}^0) \to \sigma_1 \to \alpha(\tilde{\mathbf{m}}^1)$ under the supervision of $\mathscr{P}$ in $N$. Since $\beta(\alpha(\tilde{\mathbf{m}}^1)) \to \tilde{\sigma}_2 \to \beta(\alpha(\tilde{\mathbf{m}}^2))$, it follows that $t_i$ is control-enabled under the marking $\alpha(\tilde{\mathbf{m}}^1)$ under the supervision of $\mathscr{P}$. We now show that $t_i \in T_e(\alpha(\tilde{\mathbf{m}}^1))$ in $N$. This observation is established by contradiction. If $t_i \notin T_e(\alpha(\tilde{\mathbf{m}}^1))$ in $N$, then $\exists\ p \in ({}^\bullet t_i)_N$ such that $\alpha(\tilde{\mathbf{m}}^1)\ (p) = 0$. From the definition of $\alpha(\bullet)$ we infer $\tilde{\mathbf{m}}^1(p) = 0$, and $\forall\ \hat{p} \in ((p^\bullet)^\bullet_{\tilde{N}}) \cap \hat{\Pi}$, $\tilde{\mathbf{m}}^1(\hat{p}) = 0$. By construction $({}^\bullet p)_{\tilde{N}} \cap \hat{T} = \emptyset$. Clearly, if at the marking $\tilde{\mathbf{m}}^1$ none of the transitions in the set $T - \{t_i\}$ fire, we can infer the transition $t_i$ cannot fire even once in $\tilde{N}$. This is a contradiction as $\tilde{\sigma}_2|_T = t_i$. Hence $t_i \in T_e(\alpha(\tilde{\mathbf{m}}^1))$ in $N$. Therefore, $t_i$ can fire at the marking $\alpha(\tilde{\mathbf{m}}^1)$ under the supervision of $\mathscr{P}$ in $N$. From the construction we note that $\alpha(\tilde{\mathbf{m}}^1) \to \mathbf{t}_i \to \alpha(\tilde{\mathbf{m}}^2)$. Hence the result. □

For the supervisory policy $\mathscr{P}$ introduced above, the following lemma relates the existence of a firing string under the supervision of $\mathscr{P}$ in $N$ to that of a corresponding string in $\tilde{N}$ under the supervision of $\tilde{\mathscr{P}}$.

*Lemma* 5. If $\mathbf{m}^0 \to \sigma_1 \to \mathbf{m}^1$ under the supervision of $\mathscr{P}$ in $N$ then $\beta(\mathbf{m}^0) \to \tilde{\sigma}_1 \to \beta(\mathbf{m}^1)$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$, where $\tilde{\sigma}_1|_T = \sigma_1$, and the supervisory policies $\mathscr{P}$ and $\tilde{\mathscr{P}}$ are as defined earlier.

*Proof*. This result is established by an induction argument over the length of $\sigma_1$. The base case is established by letting $\sigma_1$ equal the empty-string. As the induction hypothesis let $\mathbf{m}^0 \to \sigma_1 \to \mathbf{m}^1$ under the supervision of $\mathscr{P}$ in $N$ and $\beta(\mathbf{m}^0) \to \tilde{\sigma}_1 \to \beta(\mathbf{m}^1)$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$ such that $\tilde{\sigma}_1|_T = \sigma_1$ when the length of $\sigma_1 \leq n$ for some $n \in \mathcal{N}$.

As the induction step let $\mathbf{m}^0 \to \sigma_1 \to \mathbf{m}^1 \to t_i \to \mathbf{m}^2$ under the supervision of $\mathscr{P}$ in $N$. Since $\mathbf{m}^1 \to t_i \to \mathbf{m}^2$ under the supervision of $\mathscr{P}$ in $N$, we infer the existence of $\tilde{\sigma}_2$ such that $\beta(\mathbf{m}^1) \to \tilde{\sigma}_2 \to \tilde{\mathbf{m}}$ under the supervision of $\tilde{\mathscr{P}}$ in $\tilde{N}$ where $\tilde{\sigma}_2|_T = t_i$. From Lemma 3, we infer the existence of $\tilde{\sigma}_3$ such that $\beta(\mathbf{m}^1) \to \tilde{\sigma}_3 \to \beta(\alpha(\tilde{\mathbf{m}}))$ where $\tilde{\sigma}_3|_T = \tilde{\sigma}_2|_T = t_i$. From the construction of $\tilde{N}$ we infer $\beta(\alpha(\tilde{\mathbf{m}})) = \beta(\mathbf{m}^1)$. Hence the result. □

The following theorem establishes the fact that the supervisory policy $\mathscr{P}$ enforces liveness in $N$.

*Theorem* 1. Let $\mathscr{P}$ be the supervisory policy for the PN $N$ as defined earlier, then $\mathscr{P}$ enforces liveness in $N$.

*Proof*. From Theorem 2 (Sreenivas, 1997c), we know the supervisory policy $\tilde{\mathscr{P}}$ enforces liveness in the II-FCPN $\tilde{N}$. Suppose $\mathbf{m}^0 \to \sigma_1 \to \mathbf{m}^1$ under the supervision of $\mathscr{P}$ in $N$, then by Lemma 5 we infer $\beta(\mathbf{m}^0) \to \tilde{\sigma}_1 \to \beta(\mathbf{m}^1)$ under the supervision of $\mathscr{P}$ in $\tilde{N}$. Since $\mathscr{P}$ enforces liveness in $\tilde{N}$, $\forall t_i \in T$, $\exists \tilde{\sigma}_2 \in \tilde{T}^*$, such that $\beta(\mathbf{m}^1) \to \tilde{\sigma}_2 t_i \to \tilde{\mathbf{m}}^2$ under the supervision of $\mathscr{P}$ in $\tilde{N}$. From Lemma 4 we infer $\alpha(\beta(\mathbf{m}^1))\ (= \mathbf{m}^1) \to \tilde{\sigma}_2|_T t_i \to \alpha(\tilde{\mathbf{m}}^2)$ under the supervision of $\mathscr{P}$ in $N$. Therefore, $\mathscr{P}$ enforces liveness in $N$.

4. *Conclusions*

We considered *Petri nets* (PNs) whose *free-choice equivalent* is an *independent, increasing, free-choice Petri net* (II-FCPN) (cf. Section 4.1, Sreemniovas, 1997c). For the class of non-live, II-FCPNs there exists a readily available supervisory policy that enforces liveness. In a live PN, it is possible to fire any transition from every reachable marking, although not necessarily immediately. In this paper we showed the readily available policy that enforces liveness in the equivalent II-FCPN can be effectively converted to a similar policy for the original PN that is not necessarily an II-FCPN. As a future research direction we suggest investigations into other classes of free-choice equivalent PNs for which there is a readily available supervisory policy that enforces liveness, and an investigation into procedures that convert these policies into a corresponding policy for the original PN. We also suggest identification of structural conditions of the original net that imply the converted FCPN will be an II-FCPN as an additional future research direction.

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