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Augmented Infinitesimal Perturbation Analysis: An Alternate Explanation

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Abstract. *Augmented infinitesimal perturbation analysis* (APA) was introduced by Gaivoronski [1991] to increase the purview of the theory of Infinitesimal Perturbation Analysis (IPA). In reference [Gaivoronski 1991] it is shown that an unbiased estimate for the gradient of a class of performance measures of DEDS represented by *generalized semi-Markov processes* (GSMPs) (cf. [Glynn 1989] can be expressed as a sum of an IPA-estimate and a term that takes into account the event order changes. In this paper we present an alternate approach to establishing the result of Gaivoronski, and from this we derive a necessary and sufficient condition for the validity of the IPA algorithm for this class of performance measures. Finally we validate our results by simulation examples.

Key Words: DEDS, performance analysis, IPA

1. Introduction

It is assumed that the reader is familiar with *infinitesimal perturbation analysis* (IPA) and variants thereof. For a detailed treatment of this subject matter we refer the reader to Ho and Cao [1991], Glasserman [1991], Suri [1989]. In this paper we conceptualize a DEDS as a parametric family of stochastic processes $\{\mathbf{Z}(t, \theta), t \geq 0, \theta \in \Theta\}$ defined on an appropriate probability space, where t is the time index and Θ is the parameter space. A performance measure $L_\theta: \{\mathbf{Z}(t, \theta), t \geq 0\} \rightarrow \Re$, is a mapping that assigns to each sample path $\mathbf{Z}(t, \theta)$ a real number. The performance measure L_θ is random as $\{\mathbf{Z}(t, \theta), t \geq 0\}$ is a stochastic process. One of the main concerns in IPA is deciding the following equivalence:

$$\frac{d}{d\theta} E[L_\theta] = ? = E \left[\frac{d}{d\theta} L_\theta\right].$$

That is, we are concerned with the problem of deciding if the derivative of the average performance is the average of the derivative of the performance. This equivalence, when

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true, provides a theoretical sanction for the use of IPA and has been studied by various researchers [Glasserman 1988a, Heidelberger and Cao 1985, Gong and Ho 1987, Glasserman and Gong 1990, Gong 1987, Zazanis 1986, Glasserman and Yao 1991, 1992a, b, Cao 1985, 1987, 1988, Cao and Ho 1987, Wardi 1988]. Typically, researchers exploring IPA validity guarantee the continuity of L_θ with respect to θ under certain conditions, and after implicitly imposing some mild restrictions on the slope of L_θ, a sufficient condition for the exchange of the differentiation operator and the expectation is obtained. For further details we refer the reader to Chapter 3 of Glasserman's book [1991]. In this paper we take a different approach—while providing an alternate explanation to *augmented infinitesimal perturbation analysis* (APA) introduced by Gaivoronski, we explicitly identify a term that is to be added to the IPA-estimate of the gradient. It trivially follows that when the expected value of this newly added term is zero, we have a necessary and sufficient condition for IPA validity. The details constitute a significant portion of this paper.

In the remainder we assume the parametric family $\{\mathbf{Z}(t, \theta)\}$ is the family of possible outputs of a simulation. We assume the simulation procedure uses repeated (potentially infinite) calls to an idealized random number generator that generates independent, uniformly distributed random numbers in the unit interval. For a given value of the parameter $\theta \in \Theta$, the simulation procedure transforms the random variables returned by the ideal random number generator into a sample path in $\{\mathbf{Z}(t, \theta), t \geq 0\}$. We use the symbol $\xi \in [0, 1)^\infty$ to represent the output of the repeated calls to the idealized random number generator. For a given value of $\theta \in \Theta$, an element $\xi \in [0, 1)^\infty$ identifies an element in $\{\mathbf{Z}(t, \theta), t \geq 0\}$ and in turn identifies $L_\theta(\xi)$, the sample-path performance measure computed for the specific values of θ and ξ. For analytical simplicity we only consider scalar values for θ and we let $\Theta = \Re$.

Ho and Cao (cf. chapter 4, [1991]) observe that if the sample path performance measure $L_\theta(\xi)$ is *uniformly differentiable* with respect to θ, $\forall\ \theta \in \Re$, $\forall\ \xi \in [0, 1)^\infty$ then $(d/d\theta)E[L_\theta(\xi)] = E[(d/d\theta)L_\theta(\xi)]$. It must be noted that some sample-path performance measures are not uniformly differentiable with respect to θ and yet $(d/d\theta)E[L_\theta(\xi)] = E[(d/d\theta)L_\theta(\xi)]$, that is uniform differentiability is sufficient, but not necessary. Ho and Cao (cf. chapter 5, [1991]) and Suri [1989] cite various situations where the IPA algorithm fails. There has been a considerable effort to extend the theory of IPA to handle these situations. The enhancements to the naive IPA algorithm range from application specific modifications [Vakili and Ho 1989, Gong and Cassandras 1991, Ho and Cao 1986] to general theoretical improvements [Gong and Ho 1987, Glasserman and Gong 1990, Gong 1987, Ho and Ho 1990, Shi 1991, Brémaud and Vázquez-Abad 1991, Brémaud and Gong 1991, Fu and Hu 1991, Shi 1992]. In this sense APA introduced by Gaivoronski [1991] is yet another theoretical enhancement of IPA.

In the calculation of the IPA estimate of the gradient we essentially ignore the effect of event order changes due to an infinitesimal change in the value of the parameter θ. Often, as a pedagogical aid to the calculation of the IPA estimate, we invoke an assumption called *deterministic similarity* (cf. page 74, [Ho and Cao 1991]). This assumption suggests that for any $\theta \in \Re$ and $\xi \in [0, 1)^\infty$ there is always a neighborhood of θ in which only the *timing* of events is affected, while the *order* of events remains unaltered. While the validity of the IPA interchange condition might be apparent when the above requirement is met, it should be emphasized that for a variety of problems in spite of the lack of satisfaction of deterministic similarity, the IPA estimate of the gradient can be shown to be unbiased [Zazanis 1986, Cao 1988]. In other words, there are cases when changes in θ not only change the timings of events but also the relative order of event occurrences and yet the IPA estimate of the gradient is unbiased. In a sense it is these seemingly aberrant cases that makes the study of the validity of the interchange conditions interesting. We hope that our exposition of Gaivoronski's APA [1991] will provide a better understanding of this issue.

Section 2 of this paper contains the notation and terminology used in subsequent text. We present an alternate explanation of the APA algorithm in Section 3. Section 4 contains simulation examples that illustrate the various issues raised in Section 3. We finally conclude with directions for future research.

2. Problem Definition, Notations, and Terminology

To recapitulate, a value of the parameter $\theta \in \Re$ and $\xi \in [0, 1)^\infty$ defines a sample path of infinite length and this infinite length sample path is then used to compute a sample performance measure $L_\theta(\xi)$. In particular, consider the procedure in which a clock sample for a particular event type is generated during simulation. This typically entails picking a uniformly distributed random variable ξ_i in the interval [0, 1) and then by a suitable transformation, inverse transform method, for example (cf. Section 8.2, [Law and Kelton 1991]) a clock sample $\tau(\xi_i, \theta)$ for the ith event is computed. Clock samples created this way determine the evolution of the system both temporally and behaviorally.

To make this precise consider an M/M/1 queue represented as a *homogeneous birth-death process* over the set of integers (cf. Chapter 5, [Keilson 1979]) with a reflecting boundary at zero. Let the birth rate be λ and the death rate be μ. Essentially at the ith simulation event (assume the queue is not empty) we take two clock samples τ_λ^i and τ_μ^i chosen from an exponentially distributed ensemble with means $1/\lambda$ and $1/\mu$, respectively. If the queue is empty we only take the sample τ_λ^i, for uniformity we let $\tau_\mu^i = \infty$ in this case. At each simulation event the simulation clock is updated by $\min\{\tau_\lambda^i, \tau_\mu^i\}$ and the event that corresponds to $\min\{\tau_\lambda^i, \tau_\mu^i\}$ is scheduled to occur $\min\{\tau_\lambda^i, \tau_\mu^i\}$ time units in the future from the present. The queue size is updated accordingly. Typically the values of τ_λ^i and the finite values of τ_μ^i are obtained by the inverse transform technique from two uniformly distributed random numbers in [0, 1). Essentially the simulation routine is a mapping that takes a finite sequence of independent, uniformly distributed random numbers in [0, 1) and converts into a timed trajectory of the M/M/1 queue.

If we label arrivals in the queue by the symbol α and departures by the symbol β we can convert any finite portion of the timed trajectory of the M/M/1 queue into an event string ω. This is done by recording either α or β every time a discontinuity occurs in the timed trajectory (see for example the notion of *discrete projection* in Section 4, [Sreenivas and Krogh 1991]). The set of strings generated by an M/M/1 queue is defined by the following property: $\forall\, \hat{\omega} \in pr(\omega)$, $\#(\alpha, \hat{\omega}) \geq \#(\beta, \hat{\omega})$, where $pr(\bullet)$ denotes the prefix set, and $\#(\gamma, \bullet)$ is the number of occurrences (or *score*) of symbol γ, in the string argument. For every string $\omega \in \{\alpha, \beta\}^*$,[1] there is a set of individual clock samples that correspond to it (see for example the notion of *inverse discrete projection* in Section 4 [Sreenivas and Krogh

1991]). For example, if for a nonempty initialization of the queue we observe an arrival followed with a departure, we say we observe the string $\alpha\beta$. The set of clock samples that corresponds to this string is

$$\Upsilon(\alpha\beta) = \{\tau_\lambda^1, \tau_\mu^1, \tau_\lambda^2, \tau_\mu^2 \mid \tau_\lambda^1 < \tau_\mu^1 \text{ and } \tau_\mu^2 < \tau_\lambda^2\}.$$

The clock samples τ_λ^1, τ_μ^1, τ_λ^2 and τ_μ^2 are obtained from uniformly distributed random numbers ξ_λ^1, ξ_μ^1, ξ_λ^2, and ξ_μ^2 according to the following expressions $\tau_\lambda^i = (-1/\lambda)\ln(\xi_\lambda^i)$ and $\tau_i^\mu = (-1/\mu)\ln(\xi_\mu^i)$ for $i \in \{1, 2\}$. The set $\Upsilon(\alpha\beta)$ in turn corresponds to the set

$$\mathcal{R}(\alpha\beta, \lambda, \mu) = \{\xi_\lambda^1, \xi_\mu^1, \xi_\lambda^2, \xi_\mu^2 \mid \xi_\lambda^1 > (\xi_\mu^1)^{\lambda/\mu} \text{ and } \xi_\mu^2 > (\xi_\lambda^2)^{\mu/\lambda}\}.$$

In general, for every event-string ω, there is a corresponding subset of $[0, 1]^{\eta(\omega)}$, where $\eta(\omega)$ is the number of calls that have to be made to the idealized uniform random number generator to generate the event string ω. We denote this subset of $[0, 1)^{\eta(\omega)}$ by the symbol $\mathcal{R}(\omega, \theta)$. The dependence of $\mathcal{R}(\cdot, \cdot)$ on θ is obvious. It is useful to think of $\mathcal{R}(\cdot, \cdot)$ as the collection of elements in $[0, 1)^{\eta(\omega)}$ that preserve *deterministic similarity* (cf. page 74, [Ho and Cao 1991]). We denote the set of event-strings that can be generated by n calls to the idealized uniform random number generator by $\eta^{-1}(n)$.

Calls to the idealized uniform random number generator are only made a finite (but unbounded) number of times, so the domain $[0, 1)^\infty$ of $L_\theta:[0, 1)^\infty \to \Re$ is meaningful only in the limiting sense. For every observed event sequence $\omega \in \eta^{-1}(n)$, it is useful to envision a performance function of the form $L_{\theta,\omega}^n:[0, 1)^n \to \Re$. The conditional expectation $E[L_{\theta,\omega}^n(\xi) \mid \omega]$ is given by

$$E[L_{\theta,\omega}^n(\xi) \mid \omega] = \frac{\int_{\mathcal{R}(\omega,\theta)} L_{\theta,\omega}^n(x)\, dx}{\int_{\mathcal{R}(\omega,\theta)} dx}, \tag{1}$$

From the above equation we get

$$p(\omega) = \int_{\mathcal{R}(\omega,\theta)} dx$$

$$E[L_\theta^n(\xi)] = \sum_{\omega\in\eta^{-1}(n)} p(\omega) \frac{\int_{\mathcal{R}(\omega,\theta)} L_{\theta,\omega}^n(x)\, dx}{\int_{\mathcal{R}(\omega,\theta)} dx} = \sum_{\omega\in\eta^{-1}(n)} \int_{\mathcal{R}(\omega,\theta)} L_{\theta,\omega}^n(x)\, dx \tag{2}$$

In the next section we will develop a procedure to compute $(d/d\theta)E[L_\theta^n(\xi)]$.

3. Augmented Infinitesimal Perturbation Analysis

Differentiating Equation (2) we get

$$\frac{d}{d\theta}E[L_\theta^n(\xi)] = \sum_{\omega\in\eta^{-1}(n)} \int_{\mathcal{R}(\omega,\theta)} \frac{d}{d\theta}L_{\theta,\omega}^n(x)\,dx + \sum_{\omega\in\eta^{-1}(n)} D^n(\omega,\theta), \tag{3}$$

where $D^n(\omega, \theta)$ is defined as follows,

$$D^n(\omega,\theta) = \lim_{\Delta\theta\to 0} \frac{\int_{\mathcal{R}(\omega,\theta+\Delta\theta)} L_{\theta,\omega}^n(x)\,dx - \int_{\mathcal{R}(\omega,\theta)} L_{\theta,\omega}^n(x)\,dx}{\Delta\theta}. \tag{4}$$

We assume that $\forall\ \omega$, $\forall\ \theta$, $D^n(\omega, \theta)$ exists. For details on the above derivation we refer the reader to the original paper by Gaivoronski [1991]. It is important to point out that in the derivation of Equation (3) it is assumed that $L_{\theta,\omega}^n(x)$ is uniformly differentiable with respect to θ, while no such assumption is made on $L_\theta^n(x)$. There is a strong parallel between this and the material in Section IV of Cao's paper [1985] that deals with discontinuous performance measures.[2] However, in this paper we do not require the explicit representation of $L_\theta^n(x)$ as a sum of a function that is uniformly differentiable with respect to θ and a function that is piecewise constant with respect to θ (cf. Equation (15), [Cao 1985]). In spite of this lack of explicit structure on the performance measure, as we will see shortly, we can effectively compute the limit term in the right-hand side of Equation (22) in the paper by Cao.

Observe that the integrand in the first term of Equation (3) is the regular IPA expression. Also, it is straightforward to see that under the assumptions needed for Equation (3), a necessary and sufficient condition for IPA validity is

$$\sum_{\omega\in\eta^{-1}(n)} D^n(\omega,\theta) = 0. \tag{5}$$

This observation is similar to Theorem 2 in reference [Cao 1985]. If Equation (5) is violated, that is IPA is invalid, we will see shortly that the addition of an extra term eliminates the bias in the IPA estimate. As an illustration of Equation (5) consider the two examples described below.

EXAMPLE. For $\theta < 0.5$, let L_θ: $[0, 1) \to \mathfrak{R}$ be defined as follows,

$$L_\theta(x) = \begin{cases} x & \text{if } 0 \le x < \theta, \\ -\theta & \text{if } \theta \le x < 2\theta, \\ 0 & \text{otherwise.} \end{cases}$$

If we label the instances $0 \le \xi < \theta$ as α, $\theta \le \xi < 2\theta$ as β and $2\theta \le \xi < 1$ as γ, we obtain the following expressions for the $D^1(\cdot, \theta)$ terms.

$$D^1(\alpha, \theta) = \lim_{\Delta\theta \to 0} \frac{\int_0^{\theta+\Delta\theta} \xi \, d\xi - \int_0^{\theta} \xi \, d\xi}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0} \frac{(\theta + \Delta\theta)^2 - \theta^2}{2\,\Delta\theta} = \theta,$$

$$D^1(\beta, \theta) = \lim_{\Delta\theta \to 0} \frac{\int_{\theta+\Delta\theta}^{2\theta+2\Delta\theta} -\theta \, d\xi - \int_{\theta}^{2\theta} -\theta \, d\xi}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0} \frac{-\theta(\theta + \Delta\theta - \theta)}{\Delta\theta} = -\theta,$$

$$D^1(\gamma, \theta) = 0.$$

Therefore $\Sigma_{\omega \in \eta^{-1}(1)} D^1(\omega, \theta) = 0$. Hence we hypothesize that the IPA term should be unbiased for this performance measure. To verify this we observe that $E[L_\theta(\xi)] = -\theta^2/2$, hence $(d/d\theta)E[L_\theta(\xi)] = -\theta$. Now,

$$\frac{d}{d\theta} L_\theta(x) = \begin{cases} 0 & \text{if } 0 \le x < \theta, \\ -1 & \text{if } \theta \le x < 2\theta, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $E[(d/d\theta)L_\theta] = -\theta$. Hence the IPA estimate is unbiased as predicted earlier.

EXAMPLE. Let the parameter θ be represented as a 2-tuple $\theta = (\lambda, \mu)$ and let $L_\theta[0, 1)^2 \to \Re$ be implicitly defined by $L_\theta = \min\{\tau_\lambda, \tau_\mu\}$, where $\tau_\lambda = (-1/\lambda) \ln(\xi_\lambda)$ and $\tau_\mu = (-1/\mu) \ln(\xi_\mu)$ for $(\xi_\lambda, \xi_\mu) \in [0, 1)^2$.

For this example $E[L_\theta]$ is given by the integral

$$\int_0^1 \int_{\xi_\mu^{\lambda/\mu}}^1 \tau_\lambda \, d\xi_\lambda \, d\xi_\mu + \int_0^1 \int_{\xi_\lambda^{\mu/\lambda}}^1 d\xi_\mu \, d\xi_\lambda$$

$$= \frac{1}{\lambda + \mu} \times \frac{\lambda}{\lambda + \mu} + \frac{1}{\lambda + \mu} \times \frac{\mu}{\lambda + \mu} = \frac{1}{\lambda + \mu}.$$

So,

$$\frac{d}{d\lambda} E[L_{(\lambda,\mu)}] = \frac{-1}{(\lambda + \mu)^2}.$$

The IPA estimate is known to provide an unbiased estimate of the gradient for this example. This can be verified by observing that

$$\int_0^1 \int_{\xi_\mu^{\lambda/\mu}}^1 \frac{d}{d\lambda} \tau_\lambda \, d\xi_\lambda \, d\xi_\mu + \int_0^1 \int_{\xi_\lambda^{\mu/\lambda}}^1 \frac{d}{d\lambda} \tau_\mu \, d\xi_\mu \, d\xi_\lambda = \int_0^1 \int_{\xi_\mu^{\lambda/\mu}}^1 \frac{-1}{\lambda} \tau_\lambda \, d\xi_\lambda \, d\xi_\mu + 0$$

$$= \frac{-1}{\lambda} \times \frac{1}{\lambda + \mu} \times \frac{\lambda}{\lambda + \mu} = \frac{-1}{(\lambda + \mu)^2}.$$

Labeling instances of $\tau_\lambda < \tau_\mu$ as α and $\tau_\mu < \tau_\lambda$ as β, we hypothesize that for this example $D^2(\alpha, \theta) + D^2(\beta, \theta) = 0$. This is formally verified as follows:

$$D^2(\alpha, \theta) = \int_0^1 \left(\lim_{\xi_\lambda \to 1} \frac{1}{\lambda} \ln(\xi_\lambda) \times \frac{d}{d\lambda} 1 - \lim_{\xi_\lambda \to \xi_\mu^{\lambda/\mu}} \frac{1}{\lambda} \ln(\xi_\lambda) \times \frac{d}{d\lambda} \xi_\mu^{\lambda/\mu} \right) d\xi_\mu$$

$$= \int_0^1 \left(0 - \frac{1}{\lambda} \times \frac{\lambda}{\mu} \times \ln(\xi_\mu) \times \xi_\mu^{\lambda/\mu} \times \ln(\xi_\mu) \times \frac{1}{\mu} \right) d\xi_\mu$$

$$= \frac{-1}{\mu^2} \int_0^1 (\ln(\xi_\mu))^2 \, \xi_\mu^{\lambda/\mu} \, d\xi_\mu$$

$$= \frac{-2\mu}{(\lambda + \mu)^3}$$

and

$$D^2(\beta, \theta) = \int_0^1 \left(\lim_{\xi_\mu \to 1} \frac{1}{\mu} \ln(\xi_\mu) \times \frac{d}{d\lambda} 1 - \lim_{\xi_\mu \to \xi_\lambda^{\mu/\lambda}} \frac{1}{\mu} \ln(\xi_\mu) \frac{\mathrm{d}}{d\lambda} \xi_\lambda^{\mu/\lambda} \right) d\xi_\lambda$$

$$= \int_0^1 \left(0 - \frac{1}{\mu} \times \frac{\mu}{\lambda} \times \ln(\xi_\lambda) \times \xi_\lambda^{\mu/\lambda} \times \ln(\xi_\lambda) \times \frac{-\mu}{\lambda^2} \right) d\xi_\lambda$$

$$= \frac{\mu}{\lambda^3} \int_0^1 (\ln(\xi_\lambda))^2 \, \xi_\lambda^{\mu/\lambda} \, d\xi_\lambda$$

$$= \frac{2\mu}{(\lambda + \mu)^3}$$

Hence $D^2(\alpha, \theta) + D^2(\beta, \theta) = 0$ as hypothesized earlier.

We now call the reader's attention to the fact that the performance measure defined in the first example is discontinuous with respect to θ and yet the IPA estimate is shown to be unbiased. It should be pointed out that IPA validity of the first example can also be established by the approach advocated in Section IV of Cao's paper [1985]. In a sense, we would like to impress upon the reader that the conventional route to establishing IPA validity by first establishing continuity of the sample performance functions with respect to the parameter θ and then implicitly imposing restrictions on the slope of the sample performance function with respect to changes in θ has its limitations. The approach advocated here is to prove IPA validity by showing that $\Sigma_{\omega\in\eta^{-1}(n)} D^n(\omega, \theta) = 0$. We acknowledge the fact that the exact computation of the various $D^n(\cdot, \cdot)$ terms and the verification of Equation (5) is not easy in general. It is possible, however, to compute an unbiased estimate of $D^n(\omega, \theta)$ and this is the focus of the remainder of this section. Also, it is important to note that if Equation (5) is violated, the addition of an unbiased estimate of $D^n(\omega, \theta)$ to the IPA estimate yields an unbiased estimate of the gradient.

We now provide an alternate explanation to Gaivoronski's APA term. For a given ω, let us assume for simplicity that the set $\mathcal{R}(\omega, \theta)$ can be represented as the cartesian product of connected segments with each segment being a subset of [0, 1):

$$\mathcal{R}(\omega, \theta) = (\xi_1^{\min}(\theta, \omega), \xi_1^{\max}(\theta, \omega)) \times \cdots \times (\xi_n^{\min}(\theta, \omega), \xi_n^{\max}(\theta, \omega)), \tag{6}$$

where $n = \eta(\omega)$, the number of calls to be made to the idealized uniform random number generator to generate the string ω. This implicitly assumes the mapping that transforms the uniformly distributed random numbers from the idealized random number generator into clock samples for the various events is continuous (cf. Figures 3 and 4, chapter 5, [Ho and Cao 1991]). For reasons that will be apparent later we also suppose that this mapping is differentiable with respect to θ. In general, the set $\mathcal{R}(\omega, \theta)$ can be represented as the cartesian product of a countable union of connected segments where each segment is a subset of the unit interval [0, 1). For simplicity of presentation we do not consider the general case (cf. [Gaivoronski 1991] for the general case).

Assuming Equation (6) holds, we can be specific about the form of Equation (2).

$$E[L_\theta^n(\xi)] = \sum_{\omega\in\eta^{-1}(n)} \int_{\xi_1^{\min}(\theta)}^{\xi_1^{\max}(\theta,\omega)} \cdots \int_{\xi_n^{\min}(\theta,\omega)}^{\xi_n^{\max}(\theta,\omega)} L_{\theta,\omega}^n(\xi_i, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n.$$

Differentiating both sides with respect to θ we get

$$\frac{d}{d\theta} E[L_{\theta,\omega}^n(\xi)] = \sum_{\omega\in\eta^{-1}(n)} \int_{\xi_1^{\min}(\theta,\omega)}^{\xi_1^{\max}(\theta,\omega)} \cdots \int_{\xi_n^{\min}(\theta,\omega)}^{\xi_n^{\max}(\theta,\omega)} \frac{d}{d\theta} L_{\theta,\omega}^n(\xi_i, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n \tag{7}$$

$$+ \left(\frac{d}{d\theta}\xi_1^{\max}(\theta, \omega)\int_{\xi_2^{\min}(\theta,\omega)}^{\xi_2^{\max}(\theta,\omega)} \cdots \int_{\xi_n^{\min}(\theta,\omega)}^{\xi_n^{\max}(\theta,\omega)} L_{\theta,\omega}^n(\xi_1^{\max}(\theta, \omega), \xi_2, \ldots, \xi_n)\, d\xi_2 \ldots d\xi_n\right.$$

$$- \frac{d}{d\theta} \xi_1^{\min}(\theta, \omega) \int_{\xi_2^{\min}(\theta,\omega)}^{\xi_2^{\max}(\theta,\omega)} \cdots \int_{\xi_n^{\min}(\theta,\omega)}^{\xi_n^{\max}(\theta,\omega)} L_{\theta,\omega}^n (\xi_1^{\min}(\theta, \omega), \xi_2, \ldots, \xi_n)\, d\xi_2 \ldots d\xi_n \Bigg)$$

$$+ \cdots +$$

$$+ \Bigg(\frac{d}{d\theta} \xi_n^{max}(\theta, \omega) \int_{\xi_1^{\min}(\theta,\omega)}^{\xi_1^{\max}(\theta,\omega)} \cdots \int_{\xi_{n-1}^{\min}(\theta,\omega)}^{\xi_{n-1}^{\max}(\theta,\omega)} L_{\theta,\omega}^n(\xi_i, \ldots, \xi_n^{\max}(\theta, \omega))\, d\xi_1 \ldots d\xi_{n-1}$$

$$- \frac{d}{d\theta} \xi_n^{\min}(\theta, \omega) \int_{\xi_1^{\min}(\theta,\omega)}^{\xi_1^{\max}(\theta,\omega)} \cdots \int_{\xi_{n-1}^{\min}(\theta,\omega)}^{\xi_{n-1}^{\max}(\theta,\omega)} L_{\theta,\omega}^n (\xi_i, \ldots, \xi_n{}^{\min}(\theta, \omega)\, d\xi_2 \ldots d\xi_{n-1} \Bigg) \tag{8}$$

The portion of the above equation referred to by (8) is obtained by using the following identity from integral calculus. For any $\psi \in, \Re$, any g: $\Re^2 \rightarrow \Re$, and any F: $\Re \rightarrow \Re$ defined as

$$F(\psi) = \int_{a(\psi)}^{b(\psi)} g(x, \psi)\, dx, \tag{9}$$

the derivative of $F(\psi)$ with respect to ψ is given by

$$\frac{d}{d\psi} F(\psi) = \int_{a(\psi)}^{b(\psi)} \frac{d}{d\psi} g(x, \psi)\, dx + \left(g(b(\psi), \psi) \times \frac{d}{d\psi} b(\psi) - g(a(\psi), \psi) \times \frac{d}{d\psi} a(\psi)\right), \tag{10}$$

provided the function $a(\psi)$ and $b(\psi)$ are differentiable with respect to ψ and the function $g(x, \psi)$ is continuous with respect to x in the interval $(a(\psi), b(\psi))$. With reference to Equation (8), the analog of $a(\psi)$ and $b(\psi)$ are *assumed* to be differentiable with respect to ψ, and since these bounds identify regions where deterministic similarity holds, the continuity of the analog of $g(x, \psi)$ with respect to x in the appropiate region is also guaranteed[3] for all values of ψ (cf. assumptions on Equation (6)). In additon, the analog of $g(x, \psi)$ is assumed to be uniformly differentiable with respect to ψ for all values of x in the interval $(a(\psi), b(\psi))$. Thus justifying the first term on the right-hand-side of equation 10.

The term in Equation (8) can be rewritten as

$$\int_{\xi_1^{\min}(\theta,\omega)}^{\xi_1^{\max}(\theta,\omega)} \cdots \int_{\xi_n^{\min}(\theta,\omega)}^{\xi_n^{\max}(\theta,\omega)} \sum_{i=1}^{n} \frac{L_{\theta,\omega}^n(\ldots, \xi_i^{\max}, \ldots) \times \frac{d}{d\theta} \xi_i^{\max}(\theta, \omega) - L_{\theta,\omega}^n(\ldots, \xi_i^{\min}, \ldots) \times \frac{d}{d\theta} (\theta, \omega)}{\xi_i^{\max}(\theta, \omega) - \xi_i^{\min}(\theta, \omega)}\, d\xi_1 d\xi_2 \cdots d\xi_n. \tag{11}$$

Essentially Equations (7) and (11) suggest that a sample-path estimate of the following sum is an unbiased estimate of $(d/d\theta)\ E[L(\theta,\ \xi_1,\ \xi_2,\ \ldots,\ \xi_n)]$:

$$\frac{d}{d\theta} L^n_{\theta,\omega}(\xi_1,\ \ldots,\ \xi_n)$$

$$+\sum_{i=1}^{n} \frac{L^n_{\theta,\omega}(\ldots,\xi_i^{\max},\ldots)\ \frac{d}{d\theta}\ \xi_i^{\max}(\theta,\ \omega) - L^n_{\theta,\omega}(\ldots,\xi_i^{\min},\ldots)\ \frac{d}{d\theta}\ \xi_i^{\min}(\theta,\ \omega)}{\xi_i^{\max}(\theta,\ \omega)\ -\ \xi_i^{\min}(\theta,\ \omega)}. \tag{12}$$

This expression is identical to the expression in Equation (21) in Gaivoronski's paper [1991]. There are two major tasks to be accomplished here: (i) the computation of the $L^n_{\theta,\omega}(\ldots,\ \xi_i^{\min/\max},\ \ldots)$ terms, and (ii) the computation of $(d/d\theta)\ \xi_i^{\min/\max}(\theta,\ \omega)$ terms. In general, an on-line computation of term (i) might not be feasible and we have to resort to storing the various arguments till the end of simulation when term (i) can be effectively computed. In the computation of term (ii) it is necessary to keep track of the *triggering indicator* (cf. Section 3.3, [Ho and Cao 1991]) of the various events that occur during simulation. For the general case this can be a problem; for simulations that involve Markovian clock samples this can be worked around by using the technique of uniformization, that is, using the *standard clock algorithm* (cf. Section 7.1.1, [Ho and Cao 1991]). This will become clear when we consider a few examples in the next section. Also, since the APA sum is the sum of n random numbers it is only natural to expect the variance of this term to be a problem. For this reason we have to resort to regenerative simulations to effectively compute the average APA term with reasonable confidence. We suggest techniques for reduction of the variance of the APA estimates as a future research topic.

In the next section we consider two examples that illustrate the use of the above correction term for unbiased estimates of performance gradients.

4. Examples

We illustrate the use of the APA term by considering two examples where the conventional IPA algorithm is known to yield biased estimates. The original paper by Gaivoronski [1991] contains a simulation example of a closed, product-form network with two types of customers (cf. Figure 1, [Heidelberger et al. 1988]). Our first example is motivated by the analysis of IPA-like algorithms for birth-death processes in references (Heidelberger et al. 1988; Glasserman 1988]). We consider an M/M/1 queue that is represented as a homogeneous birth-death process on the set of positive integers with a reflecting boundary at 0. We consider regenerative cycles that start and end on the state $n = 0$. In particular we are interested in the derivative of the average regenerative cycle length with respect to changes in the birth (arrival) rate. The conventional IPA algorithm is known to provide biased estimates for this problem. We investigate the process by adding additional terms to the conventional IPA estimate to obtain an unbiased estimate as suggested by Equation (12). Glasserman [1988b] and Section 5 of Heidelberger et al. [1988] concerns the problem of obtaining the derivative of the idle, or 0-state probability with respect to arrival rate. These references dealt with inhomogeneous birth-death processes, but the issues raised here are valid for homogeneous birth-death processes as well. It should be noted that if the derivative of the average regenerative cycle length with respect to changes in birthrate is correctly estimated, the problem addressed in the above-mentioned references can be solved satisfactorily. Hence the choice of this problem as one of the illustrations of Equation (12).

The second example involves a performance measure that is dependent on the occurrence of a specific event, another situation where the conventional IPA algorithm yields a gradient of zero due to the fact that the influence of event order changes is ignored. Specifically, we take two exponentially distributed random numbers τ_λ and τ_μ with means $1/\lambda$ and $1/\mu$ respectively. If $\tau_\lambda < \tau_\mu$ the performance measure is defined to be 100, and if $\tau_\mu < \tau_\lambda$ the performance measure is defined to be -100. The derivative of the performance measure is almost surely zero. Hence the average derivative of the performance measure is zero, while the derivative of the average performance measure is not zero. We illustrate by analytical calculation and by simulation that Equation (12) does provide an unbiased estimate of the derivative of the average performance measure.

4.1. Average Length of Regenerative Period of a Homogeneous Birth-Death Process

Consider the homogeneous birth-death process on the set of positive integers with a reflecting boundary at 0 as shown in Figure 2. The birthrate is λ and μ is the death rate. This process can also be thought of as an M/M/1 queue with an arrival rate of λ and a service rate of μ. We will use the two representations interchangeably for analysis purposes. Following Heidelberger et al. [1988] and Glasserman [1988b] we define a regenerative cycle to be a trajectory of this process that starts at state 0 and returns to state 0. We are interested in computing the derivative of the average length of the regenerative cycle with respect to the birthrate λ. This is illustrated in Figure 3, where T_0 is the idle period, T_1 is the busy period, and $T = T_0 + T_1$. We are interested in the quantity $E[T] = E[T_0] + E[T_1]$. Since T_0 is an exponentially distributed random variable with rate λ it follows that $E[T_0] = 1/\lambda$. Also, we know that $E[T_1]$ is the average busy period length in an M/M/1 queue, so $E[T_1] = 1/(\mu - \lambda)$. So, $E[T] = 1/\lambda + 1/(\mu - \lambda)$, hence $(d/d\lambda)\, E[T] = 1/(\mu - \lambda)^2 - 1/\lambda^2$.

If we were to use the conventional IPA algorithm we would essentially do the following: for every birth in a regenerative cycle we keep a running-sum of the length of the clock sample for each birth and at the end of the regenerative cycle we would scale this sum by the term $-1/\lambda$ (cf. Section 3.2, [Ho and Cao 1991]). We can estimate the value of the IPA term as follows: the average number of arrivals in one busy period in an M/M/1 queue is $\lambda/(\mu - \lambda)$ and each arrival contributes an average of $1/(\mu + \lambda)$ to the running sum, the average length of the idle period is $1/\lambda$. So the average value of the above-mentioned running sum is $\lambda/(\mu - \lambda) \times 1/(\mu + \lambda) + 1/\lambda$. When this quantity is scaled by $-1/\lambda$ we get the average IPA term to be

$$\frac{-1}{\lambda}\left(\frac{\lambda}{\mu - \lambda} \times \frac{1}{\mu + \lambda} + \frac{1}{\lambda}\right) = \frac{-1}{\lambda^2} - \frac{1}{\mu^2 - \lambda^2}.$$

$\frac{\lambda}{\mu}$	Theor. Grad.	IPA Term Mean	IPA Term Var.	APA Term Mean	APA Term Var.	IPA + APA
$\frac{1.0}{10.0}$	-0.987654	-1.009080	0.000008	0.022384	0.000005	-0.986696
$\frac{1.0}{5.0}$	-0.937500	-1.040542	0.000008	0.104010	0.000003	-0.936532
$\frac{1.0}{4.0}$	-0.888889	-1.064599	0.000008	0.176588	0.000023	-0.888011
$\frac{1.0}{3.0}$	-0.750000	-1.124196	0.000010	0.367117	0.000123	-0.757079
$\frac{1.0}{2.5}$	-0.555556	-1.189093	0.000006	0.626452	0.000082	-0.562641
$\frac{1.0}{2.0}$	0	-1.333333	0.000029	1.305048	0.001301	-0.027954
$\frac{1.0}{1.6}$	1.7777778	-1.642933	0.000047	3.387627	0.022510	1.744694
$\frac{1.0}{1.4}$	5.25	-2.042619	0.000088	7.217438	0.115338	5.174818
$\frac{1.0}{1.25}$	15.0	-2.775423	0.000398	17.546053	0.681711	14.770630
$\frac{1.0}{1.1}$	98.999954	-5.810129	0.001211	108.665970	52.225552	102.855843

Figure 1. Simulation details for the homogeneous birth-death process.

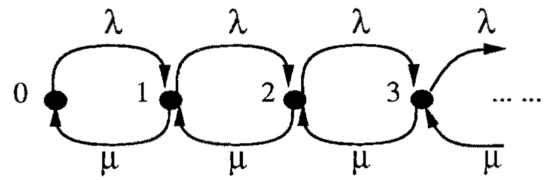

Figure 2. A homogeneous birth-death process on the set of positive integers with a reflecting boundary at 0.

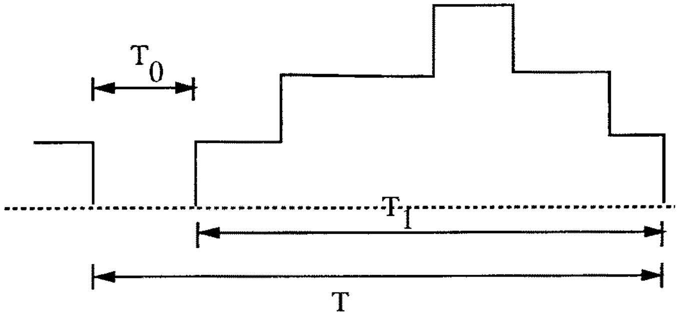

Figure 3. An illustration of the definition of the regenerative cycle.

This rough computation of the average IPA term is verified by simulation (cf. Figure 1). Clearly, the gradient as predicted by the IPA algorithm is biased.

At the ith simulation instant (assume for the present that the state is not 0) we take two exponentially distributed random numbers that correspond to the clock samples for birth and death. The shorter clock sample decides the type of event that is scheduled to occur at the next instant. In reality we take two independent, uniformly distributed random numbers ξ_λ^i and ξ_μ^i and compute two independent, exponentially distributed random clock samples τ_λ^i and τ_μ^i according to,

$$\tau_\lambda^i = \frac{-1}{\lambda} \ln(\xi_\lambda^i) \qquad \text{and} \qquad \tau_\mu^i = \frac{-1}{\mu} \ln(\xi_\mu^i).$$

If $\tau_\lambda^i < \tau_\mu^i$ then we schedule a birth (arrival) τ_λ^i time units from the current time. Likewise, if $\tau_\mu^i < \tau_\lambda^i$ then we schedule a death (departure) τ_μ^i time units from the current time.

If we ignore the past events and pay attention to the ith event type, we get the sets $\mathcal{R}(\tau_\lambda^i < \tau_\mu^i, \lambda, \mu)$ and $\mathcal{R}(\tau_\mu^i < \tau_\lambda^i, \lambda, \mu)$ to be

$$\mathcal{R}(\tau_\lambda^i < \tau_\mu^i, \lambda, \mu) = \{(\xi_\lambda^i, \xi_\mu^i) \mid (\xi_\mu^i)^{\lambda/\mu} \le \xi_\lambda^i \le 1, 0 \le \xi_\mu^i \le 1\},$$

$$\mathcal{R}(\tau_\mu^i < \tau_\lambda^i, \lambda, \mu) = \{(\xi_\lambda^i, \xi_\mu^i) \mid (\xi_\lambda^i)^{\mu/\lambda} \le \xi_\mu^i \le 1, 0 \le \xi_\lambda^i \le 1\}.$$

It is important to note that at each event instant, one of the variables ξ_λ or ξ_μ, takes on values in the entire range [0, 1), while the other takes on values over a range that is dependent on the value of the first variable. This fact will be useful to reduce the number of items in the summand for the APA term in Equation (12). Also the values of clock intervals τ_λ^i and τ_μ^i are independent of the values of past clock intervals. This would imply that the limits of integration in Equation (11) due to the ith event (note that there would be two integrals due to the ith event) are independent of the limits due to the rest of the events. At this point we call the reader's attention to the fact that the above observation can be suitably generalized for any simulation that uses Markovian clock samples that is simulated using uniformization or the *standard clock algorithm* (cf. Section 7.1.1, [Ho and Cao 1991]).

Now, if $\tau_\lambda^i < \tau_\mu^i$ then we have

$$\xi_\lambda^{i\min} = 0, \quad \xi_\lambda^{i\max} = 1 - e^{-\lambda\tau_\mu^i} \Rightarrow \frac{d}{d\lambda}\xi_\lambda^{i\min} = 0, \quad \frac{d}{d\lambda}\xi_\lambda^{i\max} = \tau_\mu^i e^{-\lambda\tau_\mu^i},$$

$$\xi_\mu^{i\min} = 0, \quad \xi_\mu^{i\max} = 1 \Rightarrow \frac{d}{d\lambda}\xi_\mu^{i\min} = 0, \quad \frac{d}{d\lambda}\xi_\mu^{i\max} = 0.$$

When computing the APA term it is necessary to compute $L(\theta, \ldots, \xi_\lambda^{i\max}, \ldots)$, where the performance index $L(\theta, \ldots)$ is the observed length of the regenerative cycle. To compute $L(\theta, \ldots, \xi_\lambda^{i\max}, \ldots)$ we add the discrepancy in the clock sample lengths $(\tau_\mu^i - \tau_\lambda^i)$ to the observed length of the regenerative cycle. We do not concern ourselves with $L(\theta, \ldots, \xi_\lambda^{i\min}, \ldots)$, $L(\theta, \ldots, \xi_\mu^{i\min}, \ldots)$, and $L(\theta, \ldots, \xi_\mu^{i\max}, \ldots)$ as these terms get multiplied by zero-valued derivatives in the APA term.

Likewise if $\tau_\mu^i < \tau_\lambda^i$ then we have

$$\xi_\mu^{i\min} = 0, \quad \xi_\mu^{i\max} = 1 - e^{-\mu\tau_\lambda^i} \Rightarrow \frac{d}{d\lambda}\xi_\mu^{i\min} = 0, \quad \frac{d}{d\lambda}\xi_\mu^{i\max} = -\tau_\lambda^i \frac{\mu}{\lambda} e^{-\mu\tau_\lambda^i},$$

$$\xi_\lambda^{i\min} = 0, \quad \xi_\lambda^{i\max} = 1 \Rightarrow \frac{d}{d\lambda}\xi_\lambda^{i\min} = 0, \quad \frac{d}{d\lambda}\xi_\lambda^{i\max} = 0,$$

and to compute $L(\theta, \ldots, \xi_\mu^{i\max}, \ldots)$ we add the term $(\tau_\lambda^i - \tau_\mu^i)$ to the observed regenerative cycle length.

If $\tau_\lambda^i < \tau_\mu^i$, we have the following contribution to the APA term,

$$L(\theta, \ldots, \xi_\lambda^{i\max}, \ldots) \times \left[\frac{\tau_\mu^i\, e^{-\lambda\tau_\mu^i}}{1 - e^{-\lambda\tau_\mu^i}} \right], \tag{13}$$

and if $\tau_\mu^i < \tau_\lambda^i$, and we have the following contribution to the APA term,

$$L(\theta, \ldots, \xi_\lambda^{i\max}, \ldots) \times \left[\frac{-\tau_\lambda^i\,(\mu/\lambda)\; ee^{-\mu\tau_\lambda^i}}{1 - e^{-\mu\tau_\lambda^i}} \right]. \tag{14}$$

The table in Figure 1 shows the derivative of the average regenerative cycle length with respect to changes in the birthrate for various choices of the load (i.e., the ratio λ/μ). The columns in this table are self-explanatory. For each load the homogeneous birth-death process was simulated for 75,000 regenerative cycles and repeated 10 times with different initial seeds. The variance numbers in the table were computed for the 10 repetitions. As can be seen the correction term suggested in equation (12) does provide an unbiased estimate of the gradient.

4.2. A Counting Process Example

Consider the following counting process: we take two exponentially distributed random numbers τ_λ and τ_μ, where the first random number has a mean of $1/\lambda$ and the second a mean of $1/\mu$. We define a performance measure $L(\lambda, \mu)$ as follows:

$$L(\lambda, \mu) = \begin{cases} 100 & \text{if } \tau_\lambda < \tau_\mu, \\ -100 & \text{if } \tau_\mu < \tau_\lambda. \end{cases}$$

It is straightforward to see that

$$E[L(\lambda, \mu)] = 100\,\frac{\lambda - \mu}{\lambda + \mu} \quad \text{and} \quad \frac{d}{d\theta} E[L(\lambda, \mu)] = 200\,\frac{\mu}{(\lambda + \mu)^2}.$$

Following an argument that is similar to one in the previous section we obtain the following algorithm: if $\tau_\lambda < \tau_\mu$ the APA term is

$$100 \times \frac{\tau_\mu \times e^{-\lambda\tau_\mu}}{1 - e^{-\lambda\tau_\mu}}, \tag{15}$$

and if $\tau_\mu < \tau_\lambda$ the APA term is

$$100 \times \frac{\tau_\lambda \times \mu/\lambda\; e^{-\mu\tau_\lambda}}{1 - e^{-\mu\tau_\lambda}}. \tag{16}$$

The expected value of the APA term is given by the integral

$$\int_0^\infty \int_0^{\tau_\mu} 100 \times \frac{\tau_\mu e^{-\lambda\tau_\mu}}{1 - e^{-\lambda\tau_\mu}} \times \lambda e^{-\lambda\tau_\lambda} \times \mu e^{-\mu\tau_\mu}\, d\tau_\lambda\, d\tau_\mu$$

$$+ \int_0^\infty \int_0^{\tau_\lambda} 100 \times \frac{\tau_\lambda\, \mu/\lambda\; e^{-\mu\tau_\lambda}}{1 - e^{-\mu\tau_\lambda}} \times \lambda e^{-\lambda\tau_\lambda} \times \mu e^{-\mu\tau_\mu}\, d\tau_\mu\, d\tau_\lambda$$

$$= \frac{100\mu}{(\lambda + \mu)^2} + \frac{100\mu}{(\lambda + \mu)^2} = \frac{200\mu}{(\lambda + \mu)^2}.$$

For this problem it can be shown that the IPA estimate for the gradient is zero, hence E[IPA term + APA term] = E(APA term] = $200\mu/(\lambda + \mu)^2$. Hence Equation (12) does provide an unbiased estimate for this problem.

The table in Figure 4 illustrates the derivative of the performance measure defined above for various choices of the ratio λ/μ. For each value of λ/μ the APA term was computed for 100,000 trials and this experiment was repeated 10 times. The variance numbers in the table were computed for the 10 repetitions. As evidenced by the simulation data, Equation (12) does provide an unbiased estimate of the derivative of the performance measure for this example.

$\frac{\lambda}{\mu}$	Theor. Grad.	APA Term Mean	Var.
$\frac{1.0}{10.0}$	16.528925	16.573528	0.003280
$\frac{1.0}{5.0}$	27.777779	27.818027	0.005002
$\frac{1.0}{3.0}$	37.500000	37.537273	0.004619
$\frac{1.0}{2.5}$	40.816326	40.848961	0.003387
$\frac{1.0}{2.0}$	44.444443	44.468376	0.003585
$\frac{1.0}{1.6}$	47.337276	47.356056	0.003305
$\frac{1.0}{1.4}$	48.611111	48.620918	0.003377
$\frac{1.0}{1.25}$	49.382717	49.386917	0.003219
$\frac{1.0}{1.1}$	49.886620	49.885071	0.003637

Figure 4. Simulation details for the counting example.

5. Conclusions

In a recent paper Gaivoronski [1991] introduced a new variant of IPA called *augmented infinitesimal perturbation analysis* (APA) where it was shown that any bias in the IPA estimate of the gradient of a class of performance measures can be removed by adding an extra term that takes into account the influence of event order changes. In this paper we provide an alternate explanation for Gaivoronski's APA. Using this result we identify a necessary and sufficient condition for the validity of the IPA algorithm for this class of performance measures. We provide two examples that provide an empirical validation of our interpretation of the results in the above-mentioned reference.

As a future research direction we suggest investigations into methods for the reduction in variance of the newly added APA term. Also, we call the reader's attention to the relevance of the observations in this paper to sample-path based estimation of derivatives of higher order. We believe a creative and judicious use of these higher-order derivatives holds promise in the estimation of response surfaces for use in optimization of DEDS.

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Notes

1. We assume familiarity with the conventions regarding finite representations of languages on a given symbol set. For details we refer the reader to Section 1.9 of Lewis and Papadimitriou's book [1986].
2. We thank the second referee for bringing this similarity to our attention.
3. We remind the reader that we restrict our attention to performance measures expressed as an integral over time of the state trajectory of the GSMP (cf. Equations (3) and (4), page 16, chapter 2, [Ho and Cao 1991].

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