On a Decidable Class of Partially Controlled Petri Nets With Liveness Enforcing Supervisory Policies

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***Abstract*—We identify a class of partially controlled *Petri net* (PN) structures, which is denoted by $\mathcal{G}$, that strictly includes the class of partially controlled *free-choice* (FC) PN structures. We show that there is a supervisory policy that enforces liveness in an arbitrary instance $N(\mathbf{m^0})$, where $N \in \mathcal{G}$, if and only if there is a similar policy for an FCPN that results when the construction procedure enunciated in this paper is executed with N and its controllable transition set as input. Since the existence of a supervisory policy in an arbitrary partially controlled FCPN is decidable, it follows that the existence of similar policies for any $N(\mathbf{m^0})$, where $N \in \mathcal{G}$, is also decidable. Furthermore, when it exists, the minimally restrictive supervisory policy that enforces in a member of $\mathcal{G}$ is characterized by a *right-closed set* of markings.**

***Index Terms*—Discrete event systems, Petri nets (PNs), supervisory control .**

I. Introduction

A *liveness* specification is characterized as a property that certifies that, irrespective of past executions, something desirable can still occur in a concurrent system [2]. In this paper, we concern ourselves with *Petri net* (PN) [3] models of concurrent systems, where transitions in the PN represent activities. The liveness property we seek guarantees that, irrespective of the past transition firings, every transition is potentially fireable, although not immediately necessary, in the future. A concurrent system with this property does not experience livelocks, which is a desirable property.

We consider PN models that do not meet the aforementioned liveness specification, and we explore the existence of *supervisory policies* that enforce this property. At a given marking of the PN, the supervisory policy selectively disables a subset of *controllable* transitions to enforce the desired liveness property. A PN is said to be partially controlled if the set of controllable transitions is a strict subset of the set of transitions. The existence of a liveness enforcing supervisory policy (LESP) in an arbitrary partially controlled PN is undecidable (cf. [4, Corollary 5.2]). Furthermore, neither the existence nor the nonexistence of a supervisory policy that enforces liveness in an arbitrary partially controllable PN is semidecidable (cf. [1, Ths. 3.1 and 3.2]). Therefore, we must restrict our attention to concurrent systems modeled by classes of PNs where the aforementioned problem is decidable. The rest of this paper is about a class of PNs, i.e., $\mathcal{G}$, that meets this desideratum.

Every arc from a place to a transition in a *free-choice* (FC) PN structure is either the unique input arc to the transition or is the unique output arc from the place. We identify a class of partially controlled PN structures $\mathcal{G}$ that strictly includes the class of FCPN structures. We show that an arbitrary member of $\mathcal{G}$ can be converted into an FC structure, using the construction procedure identified in this paper. We show that there is a supervisory policy that enforces liveness in the PN $N(\mathbf{m}^0)$, where $N \in \mathcal{G}$, if and only if a similar policy exists for the FCPN that results when this construction procedure is used with N and its controllable transition set as input. Since the existence of a LESP in an arbitrary partially controlled FCPN is decidable [1], it follows that the existence of a supervisory policy for any $N(\mathbf{m}^0)$, where $N \in \mathcal{G}$, is also decidable. Additionally, we characterize the minimally restrictive supervisory policy that enforces liveness in those instances for which there is a supervisory policy that enforces liveness.

The rest of this paper is organized as follows: In Section II, we present the relevant notations and definitions that are used in subsequent text, along with some observations and a brief review of the prior work in the supervisory control of PNs with an eye toward enforcing liveness constraints. The main results of this paper are in Section III. Section IV contains the conclusions.

II. Notations and Definitions and Some Preliminary Observations

A *PN structure* $N = (\Pi, T, \Phi)$ is an ordered 3-tuple, where $\Pi = \{p_1, \ldots, p_n\}$ is a set of n *places*, $T = \{t_1, \ldots, t_m\}$ is a collection of m *transitions*, and $\Phi \subseteq (\Pi \times T) \cup (T \times \Pi)$ is a set of *arcs*. The *initial marking function* (or the *initial marking*) of a PN structure N is a function $\mathbf{m}^0 : \Pi \rightarrow \mathcal{N}$, where $\mathcal{N}$ is the set of nonnegative integers. We will use the term *PN* to denote a PN structure along with its initial marking $\mathbf{m}^0$ and is denoted by the symbol $N(\mathbf{m}^0)$ [5], [3].

The *marking* of a PN N, i.e., $\mathbf{m}^i : \Pi \rightarrow \mathcal{N}$, identifies the number of *tokens* in each place. For a given marking $\mathbf{m}^i$, a transition $t \in T$ is said to be *enabled* if $\forall p \in ({}^\bullet t)_N, \mathbf{m}^i(p) \geq 1$, where $({}^\bullet x)_N := \{y \mid (y, x) \in \Phi\}$. Similarly, $(x^\bullet)_N := \{y \mid (x, y) \in \Phi\}$. The subscript N is dropped whenever the identity of the PN structure is clear. If $X \subseteq \Pi$ or $X \subseteq T$, then ${}^\bullet X = \cup_{x \in X} {}^\bullet x$ and $X^\bullet = \cup_{x \in X} x^\bullet$.

The set of enabled transitions at marking $\mathbf{m}^i$ is denoted by the symbol $T_e(N, \mathbf{m}^i)$. An enabled transition $t \in T_e(N, \mathbf{m}^i)$ can *fire*, which changes the marking $\mathbf{m}^i$ to $\mathbf{m}^{i+1}$ according to the expression $\mathbf{m}^{i+1}(p) = \mathbf{m}^i(p) - \text{card}(p^\bullet \cap \{t\}) + \text{card}({}^\bullet p \cap \{t\})$, where the symbol card($\bullet$) is used to denote the cardinality of the set argument. We say two transitions $t_1, t_2 \in T$ are in *conflict* if they share a common input place (i.e., ${}^\bullet t_1 \cap^\bullet t_2 \neq \emptyset$). In this paper, we do not consider simultaneous firing of multiple transitions.

A string of transitions $\sigma = t_1 t_2 \cdots t_k$, where $t_j \in T$ $(j \in \{1, 2, \ldots, k\})$, is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$ if 1) the transition $t_1 \in T_e(N, \mathbf{m}^i)$ and 2) for $j \in \{1, 2, \ldots, k\}$, the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ is enabled. If $\mathbf{m}^{i+k}$ results from the firing of $\sigma \in T^*$ starting from the initial marking $\mathbf{m}^i$, we represent it symbolically as $\mathbf{m}^i \rightarrow \sigma \rightarrow \mathbf{m}^{i+k}$. Given an initial marking $\mathbf{m}^0$, the set of *reachable markings* for $\mathbf{m}^0$, which is denoted by $\Re(N, \mathbf{m}^0)$, is defined as the set of markings generated by all valid firing strings starting with marking $\mathbf{m}^0$ in the PN N. A PN $N(\mathbf{m}^0)$ is said to be *live* if $\forall t \in T, \forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i)$ such that $t \in T_e(N, \mathbf{m}^j)$ (cf. *level 4 liveness*, [5], [3]).

A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \geq \mathbf{m}^1) \Rightarrow (\mathbf{m}^2 \in \mathcal{M}))$. Every right-closed set of vectors $\mathcal{M} \subseteq \mathcal{N}^n$ contains a finite set of minimal elements, i.e., $\min(\mathcal{M}) \subset \mathcal{M}$, such that 1) $\forall \mathbf{m}^1 \in \mathcal{M}, \exists \mathbf{m}^2 \in \min(\mathcal{M})$ such that $\mathbf{m}^1 \geq \mathbf{m}^2$, and 2) if $\exists \mathbf{m}^1 \in \mathcal{M}, \exists \mathbf{m}^2 \in \min(\mathcal{M})$ such that $\mathbf{m}^2 \geq \mathbf{m}^1$, then $\mathbf{m}^1 = \mathbf{m}^2$. A PN structure $N = (\Pi, T, \Phi)$ is *FC* if

$$\forall p \in \Pi,\ (\text{card}(p^\bullet) > 1) \Rightarrow (({}^\bullet(p^\bullet)) = \{p\}) . \tag{1}$$

In words, a PN structure is FC if and only if an arc from a place to a transition is either the unique output arc from that place or is the

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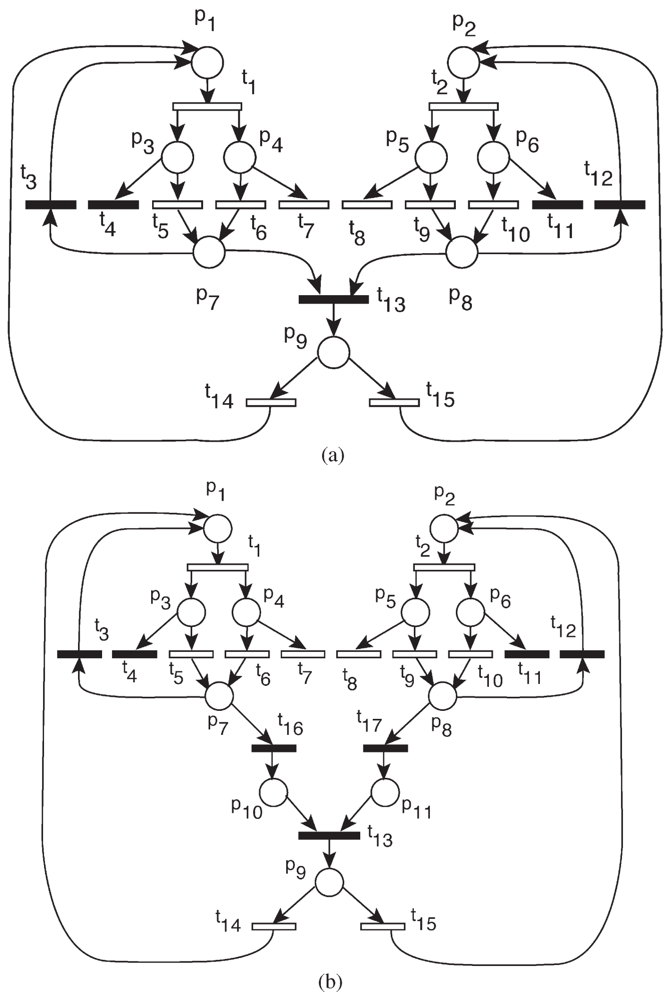

Fig. 1. PN structure $N_1$is not FC as $({}^\bullet(p_7^\bullet)) = ({}^\bullet(p_8^\bullet)) = \{p_7, p_8\}$. The PN structure N_2 is FC, as $\forall p \in \Pi_2, ({}^\bullet(p^\bullet)) = \{p\}$. The transition t_5 (t_{16}) is uncontrollable (controllable) in $N_1(N_2)$.

unique input arc to the transition. The PN N_1 shown in Fig. 1(a) is not FC. This is because card$((p_7^\bullet)_{N_1}) > 1$, card$((p_8^\bullet)_{N_1}) > 1$, and $({}^\bullet(p_7^\bullet)_{N_1})_{N_1} = ({}^\bullet(p_8^\bullet)_{N_1})_{N_1} = \{p_7, p_8\}$. On the other hand, the PN N_2 shown in Fig. 1(b) is FC (note, $({}^\bullet(p_7^\bullet)_{N_2})_{N_2} = \{p_7\}$ and $({}^\bullet(p_8^\bullet)_{N_2})_{N_2} = \{p_8\}$). A PN $N(\mathbf{m}^0)$, where $N = (\Pi, T, \Phi)$ is FC, is said to be an FCPN.

We assume that a subset of transitions, which are called *controllable transitions*, denoted by $T_c \subseteq T$ can be prevented from firing by an external agent called the supervisor. The set of *uncontrollable transitions* denoted by $T_u \subseteq T$ is given by $T_u = T - T_c$. A PN structure N that satisfies the requirement, i.e., $T_u = \emptyset$, is said to be *fully controlled*; otherwise, it is said to be *partially controlled*. In the graphic representation of PNs, controllable (uncontrollable) transitions are represented by filled (unfilled) rectangles.

A *supervisory policy* $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$ is a function that returns a 0 or a 1 for each transition and each reachable marking. The supervisory policy $\mathcal{P}$ permits the firing of transition t_j at marking $\mathbf{m}^i$ only if $\mathcal{P}(\mathbf{m}^i, t_j) = 1$. If $t_j \in T_e(N, \mathbf{m}^i)$ for some marking $\mathbf{m}^i$, we say that the transition t_j is *state enabled* at $\mathbf{m}^i$. If $\mathcal{P}(\mathbf{m}^i, t_j) = 1$, we say that the transition t_j is *control enabled* at $\mathbf{m}^i$. A transition has to be state and control enabled before it can fire. The fact that uncontrollable transitions cannot be prevented from firing by the supervisory policy is captured by the requirement that $\forall \mathbf{m}^i \in \mathcal{N}^n, \mathcal{P}(\mathbf{m}^i, t_j) = 1$, if $t_j \in T_u$. This is implicitly assumed of any supervisory policy in this paper.

A set of markings $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *control invariant* with respect to a partially controlled PN structure $N = (\Pi, T, \Phi)$ if $\mathcal{M} = \Gamma(\mathcal{M})$, where $\Gamma(\mathcal{M}) = \{\mathbf{m}^i \in \mathcal{N}^n \mid \exists \sigma \in T_u^*, \exists \mathbf{m}^j \in \mathcal{M}$, such that $\mathbf{m}^j \to \sigma \to \mathbf{m}^i\}$. Note that $\mathcal{M} \subseteq \Gamma(\mathcal{M})$, in general. If $\mathcal{M}$ is not control invariant, then $\exists \mathbf{m}^i \in \mathcal{M}, \exists t_u \in T_u, \mathbf{m}^i \to t_u \to \mathbf{m}^j$ and $\mathbf{m}^j \notin \mathcal{M}$.

A string of transitions $\sigma = t_1 t_2 \cdots t_k$, where $t_j \in T$ ($j \in \{1, 2, \ldots, k\}$), is said to be a *valid firing string* starting from the marking $\mathbf{m}^i$ under the supervision of $\mathcal{P}$ if 1) $t_1 \in T_e(N, \mathbf{m}^i), \mathcal{P}(\mathbf{m}^i, t_1) = 1$, and 2) for $j \in \{1, 2, \ldots, k\}$, the firing of the transition t_j produces a marking $\mathbf{m}^{i+j}$ and $t_{j+1} \in T_e(N, \mathbf{m}^{i+j})$ and $\mathcal{P}(\mathbf{m}^{i+j}, t_{j+1}) = 1$. The set of reachable markings under the supervision of $\mathcal{P}$ in N from the initial marking $\mathbf{m}^0$ is denoted by $\Re(N, \mathbf{m}^0, \mathcal{P})$.

A supervisory policy $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$ is said to be *marking monotone* if $\forall t \in T, \forall \{\mathbf{m}^j, \mathbf{m}^i\} \subseteq \mathcal{N}^n, (\mathbf{m}^j \geq \mathbf{m}^i) \Rightarrow (\mathcal{P}(\mathbf{m}^j, t) \geq \mathcal{P}(\mathbf{m}^i, t))$. That is, if a transition is control enabled at some marking by a marking monotone policy, it remains control enabled for all larger markings.A transition t_k is *live* under the supervision of $\mathcal{P}$ if $\forall \mathbf{m}^i \in \Re(N, \mathbf{m}^0, \mathcal{P}), \exists \mathbf{m}^j \in \Re(N, \mathbf{m}^i, \mathcal{P})$ such that $t_k \in T_e(N, \mathbf{m}^j)$ and $\mathcal{P}(\mathbf{m}^j, t_k) = 1$. A supervisory policy $\mathcal{P}$ enforces liveness if all transitions in N are live under $\mathcal{P}$. The policy $\mathcal{P}$ is said to be *minimally restrictive* if, for every supervisory policy $\widehat{\mathcal{P}} : \mathcal{N}^n \times T \to \{0, 1\}$ that enforces liveness in N, the following condition holds: $\forall \mathbf{m}^i \in \mathcal{N}^n, \forall t \in T, \mathcal{P}(\mathbf{m}^i, t) \geq \widehat{\mathcal{P}}(\mathbf{m}^i, t)$. Alternately, if a minimally restrictive supervisory policy $\mathcal{P}$ that enforces liveness in N prevents the occurrence of transition $t \in T$ at some marking $\mathbf{m}^i \in \mathcal{N}^n$, then every policy that enforces liveness in N should prevent the occurrence of $t \in T$ for the marking $\mathbf{m}^i$. There is a unique minimally restrictive policy that enforces liveness in every PN N that has some policy that enforces liveness (cf. [4, Th. 6.1]).

In general, the literature on liveness enforcement in PN models is quite vast. We review results in the literature related to liveness enforcement via supervisory control of PNs in the succeeding section.

A. Review of Relevant Results From the Literature

Monitors are places added to an existing PN structure, whose token load at any instant indicates the amount of a particular resource that is available for consumption. The input and output arcs to this place appropriately capture the consumption and production of resources in the original PN. These were originally introduced into supervisory control of PNs by Giua [6] to handle mutual exclusion constraints. Moody and Antsaklis considered *monitor-based supervisors* that enforce liveness in certain classes of PNs, where the liveness constraints can be expressed as linear inequalities, which are then implemented using monitor places [7]. This work was extended by Iordache and Antsaklis to include a sufficient condition for the existence of policies that enforce liveness in a class of PNs called *asymmetric-choice PNs*[1] [8]. Reveliotis [9] and Reveliotis *et al.* [10] addressed the problem of enforcing liveness by supervision for a class of PNs that models resource allocation systems. Some of these results can be extended to other PN classes. Ghaffari *et al.* [11] used the theory of regions to obtain a *minimally restrictive* supervisory policy that enforces liveness for a class of PNs.

The following result identifies a necessary and sufficient condition for the existence of a supervisory policy that enforces liveness in an arbitrary PN.

Theorem 2.1: (See [4, Th. 5.1]) Let $N = (\Pi, T, \Phi)$ be an arbitrary PN. There is a supervisory policy $\mathcal{P} : \mathcal{N}^n \times T \to \{0, 1\}$ that enforces liveness in $N(\mathbf{m}^0)$ if and only if there is a control-invariant subset $\mathcal{M} \subseteq \Re(N(\mathbf{m}^0))$ such that

1) $\mathbf{m}^0 \in \mathcal{M}$,

[1] [5, cf. p. 554] for a formal definition.

2) $\forall \mathbf{m}^1 \in \mathcal{M}, \exists \mathbf{m}^2, \mathbf{m}^3 \in \mathcal{M}, \exists \sigma_1, \sigma_2 \in T^*$, such that $\mathbf{m}^1 \to \sigma_1 \to \mathbf{m}^2 \to \sigma_2 \to \mathbf{m}^3$,
   a) $\mathbf{m}^3 \geq \mathbf{m}^2$,
   b) $\mathbf{x}(\sigma_2) \geq \mathbf{1}$ (i.e., all members of T appear at least once in σ_2), and
   c) $\forall \sigma_3 \in \mathrm{pr}(\sigma_1\sigma_2), (\mathbf{m}^1 \to \sigma_3 \to \mathbf{m}^4) \Rightarrow (\mathbf{m}^4 \in \mathcal{M})$,
   where $\mathbf{x}(\bullet)$ (pr($\bullet$)) is the *Parikh vector*[2] (*prefix set*) of the string argument.

Testing these conditions for an an arbitrary PN is undecidable [4]. On the other hand, it is possible to decide if there is a supervisory policy that enforces liveness in an FCPN $N(\mathbf{m}^0)$. Furthermore, the minimally restrictive supervisory policy that enforces liveness in an arbitrary FCPN is marking monotone and is characterized by a right-closed control-invariant set of markings [1, cf. $\Delta(N)$, eq. (2)].

The set of initial markings, i.e., $\Delta(N)$, for which there is a supervisory policy that enforces liveness for a PN structure N, is defined as

$$\Delta(N) = \left\{\mathbf{m}^0 | \ \exists \text{ a liveness enforcing supervisory policy for } N(\mathbf{m}^0)\right\}. \quad (2)$$

The set $\Delta(N)$ is control invariant with respect to the PN structure N. That is, if $\mathbf{m}^1 \in \Delta(N), t_u \in T_e(N, \mathbf{m}^1) \cap T_u$ and $\mathbf{m}^1 \to t_u \to \mathbf{m}^2$ in N, then $\mathbf{m}^2 \in \Delta(N)$. In words, if an uncontrollable transition t_u can fire at a marking $\mathbf{m}^1$ in $\Delta(N)$, resulting in a marking $\mathbf{m}^2$, then $\mathbf{m}^2$ is also in $\Delta(N)$. Alternately, only the firing of a controllable transition at a marking in $\Delta(N)$ can result in a new marking that is not in $\Delta(N)$. This observation directly follows from the definition of $\Delta(N)$.

Suppose $\mathbf{m}^0 \in \Delta(N)$, then the supervisory policy that control disables any (controllable) transition at a marking in $\Delta(N)$, if its firing would result in a new marking that is not in $\Delta(N)$, is the minimally restrictive LESP for $N(\mathbf{m}^0)$ [1].

If the PN structure N is fully controlled (i.e., $T_u = \emptyset$) or if N belongs to the class $\mathcal{F}$ defined in [12] (or if N belongs to the class $\mathcal{G}$ defined in Section III of this paper), the set $\Delta(N)$ is right-closed and is characterized by its minimal elements $\min(\Delta(N))$.

There is a procedure to test the control invariance of a right-closed set of markings Ψ of a PN structure N. If Ψ does not pass this test, then it is possible to find the largest subset of Ψ that is control invariant with respect to N (cf. [1, Lemma 5.10]).

If 1) N is a PN structure where $\Delta(N)$ is known to be right-closed, 2) Ψ is a right-closed set of markings that is control invariant with respect to N, 3) $\mathcal{P}_\Psi$ is a supervisory policy that control disables any (controllable) transition at a marking in Ψ if its firing would result in a new marking that is not in Ψ, and 4) $\mathbf{m}^0 \in \Psi$, we can construct the coverability graph, i.e., $G(N(\mathbf{m}^0), \mathcal{P}_\Psi)$, of $N(\mathbf{m}^0)$ under the supervision of $\mathcal{P}_\Psi$, along the same lines as the coverability graph of a PN (cf. [3, Sec. 4.2.1]).

The policy $\mathcal{P}_\Psi$ enforces liveness in $N(\mathbf{m}^0)$ if and only if

1) $\mathbf{m}^0 \in \Psi$, and
2) there is a closed-path $v \to \sigma \to v$ in $G(N(\mathbf{m}^i), \mathcal{P}_\Psi)$ ($\sigma \in T^*$), for each $\mathbf{m}^i \in \min(\Psi)$ where
   a) all transitions appear at least once in σ (i.e., $\mathbf{x}(\sigma) \geq \mathbf{1}$), and
   b) the net change in the token load in each place after the firing of σ is nonnegative (i.e., $\mathbf{C}\mathbf{x}(\sigma) \geq \mathbf{0}$).

The algorithm for the synthesis of an LESP for a PN structure N that belongs to a class where $\Delta(N)$ is known to be right-closed essentially involves a search for a right-closed set of markings Ψ that is control invariant with respect to N, where each member of $\min(\Psi)$ meets the path requirement on its coverability graph described earlier. This is done in an iterative manner starting with an initial set

$$\Psi_0 = \left\{\mathbf{m}^0 | \exists \text{ an LESP for } N(\mathbf{m}^0) \text{ if all transitions in } N \text{ are controllable}\right\}$$

which is known to be right-closed (cf. [4] and [1]). The LESP synthesis procedure is described here.

- If $\mathbf{m}^0 \notin \Psi_i$, the procedure terminates with the conclusion that there is no LESP for $N(\mathbf{m}^0)$.
- If $\mathbf{m}^0 \in \Psi_i$ and Ψ_i is not control invariant with respect to the PN structure N, it is replaced by its largest control invariant subset, i.e., Ψ_{i+1}, where $\Psi_{i+1} \subset \Psi_i$. Following this, the process is repeated with $\Psi_i \leftarrow \Psi_{i+1}$.
- If $\mathbf{m}^0 \in \Psi_i$ and Ψ_i is control invariant with respect to the PN structure N, each minimal element of the control invariant right-closed set Ψ_i is tested for the path requirement on its coverability graph described earlier.
- If all minimal elements satisfy this requirement, then the members of $\min(\Psi_i)$ are presented as a description of the LESP for $N(\mathbf{m}^0)$.
- If there are minimal elements that do not meet the path requirement, then each minimal element $\mathbf{m}^i$ that fails the requirement is "elevated" by card(Π)-many unit vectors as follows:

$$\mathbf{m}^i \leftarrow \left\{\mathbf{m}^i + \mathbf{1}_i | i \in \{1, 2, \ldots, \text{card}(\Pi)\}\right\}$$

where $\mathbf{1}_i$ is the ith unit vector. That is, the preceding process replaces the minimal element $\mathbf{m}^i$ with card(Π)-many minimal elements, which, in turn, defines a right-closed set $\Psi_{i+1} \subset \Psi_i$. After this, the process is repeated with $\Psi_i \leftarrow \Psi_{i+1}$.

This procedure forms the corpus of the algorithm used to synthesize the minimally restrictive LESP for $N(\mathbf{m}^0)$, when it exists, for a structure N for which it is known that $\Delta(N)$ is right-closed, and has been implemented in $C/C++$ on Mac (Windows) platforms using the *Xcode* (*Visual Studio 2012*) complier [13], [14]. The implementation described in [13] and [14] can be used to compute the minimally restrictive supervisory policy that enforces liveness in any member of the class of PNs $\mathcal{G}$ identified in the succeeding section.

III. Main Results

Consider a family of PN structures, which is denoted by $\mathcal{G}$, which is defined as follows: A PN $N = (\Pi, T, \Phi)$ with a set of uncontrollable transitions $T_u \subseteq T$ belongs to the family $\mathcal{G}$ if and only if $\forall t_u \in T_u, \forall t \in T - \{t_u\}$, i.e.,

$$({}^\bullet t \cap {}^\bullet t_u \neq \emptyset) \Rightarrow (\text{card}({}^\bullet t) = \text{card}({}^\bullet t_u) = 1). \quad (3)$$

That is, a PN structure N belongs to the family $\mathcal{G}$ if and only if every place in N that violates the FC requirement of (1) has output transitions that are controllable.

The PN structure N_1 shown in Fig. 1(a) meets the requirement of (3). For instance, places p_7 and p_8 are the only places in this PN structure that violate the FC requirement of (1), but all output transitions of these places are controllable (i.e., $(p_7^\bullet)_{N_1} = \{t_3, t_{13}\} \subseteq T_{c_1}$, and $(p_8^\bullet)_{N_1} = \{t_{12}, t_{13}\} \subseteq T_{c_1}$). Hence, $N_1 \in \mathcal{G}$. The PN structure N_3 shown in Fig. 2(a) is not a member of $\mathcal{G}$, as places p_1 and p_2 in N_3 violate the FC requirement of (1), and $t_1 \in (p_1^\bullet)_{N_3} \cap T_{u_3}$ and $t_3 \in (p_2^\bullet)_{N_3} \cap T_{u_3}$.

Suppose $N_1 = (\Pi_1, T_1, \Phi_1)$, $N_1 \in \mathcal{G}$, and T_{c_1} (T_{u_1}) is the set of controllable (uncontrollable) transitions of N_1. The procedure shown

[2] That is, the ith entry of $\mathbf{x}(\sigma)$, where $\sigma \in T^*$, corresponds to the number of occurrences of t_i in the string σ.

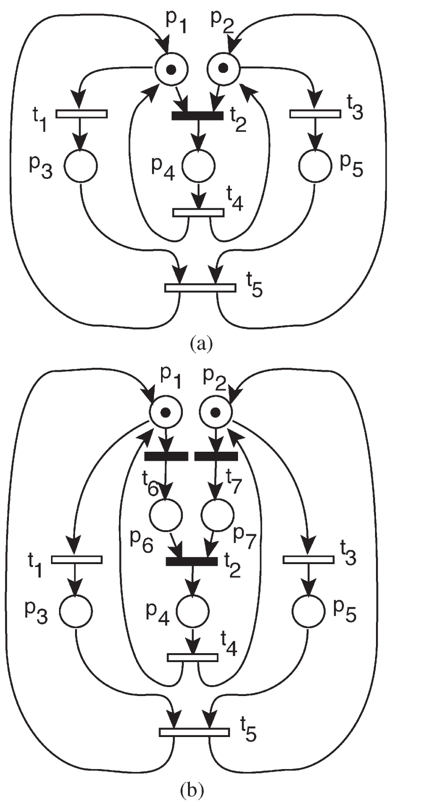

Fig. 2. PN structure N_3 is not FC as $({}^\bullet(p_1^\bullet)_{N_i})_{N_i} = \{p_1, p_2\} \neq \{p_1\}$. In addition, $N_3 \notin \mathcal{G}$, as $(p_1^\bullet)_{N_3} = \{t_1, t_2\}$ and $t_1 \in T_{u_3}$. The FC structure N_4 results when the procedure in Fig. 3 is executed with N_1 and T_{c_1} as input (and line 1 of the procedure in Fig. 3 is disabled). There is a supervisory policy that enforces liveness in $N_3(\mathbf{m}_3^0)$, and there is no liveness enforcing policy for the FCPN $N_4(\mathbf{m}_4^0)$.

convert-to-FCPN(PN structure N_1, Controllable Transition set T_{c_1})

1: **if** $N_1 \in \mathcal{G}$ (cf. equation 3) **then**
2: Create $\Pi_2, T_2, \Phi_2, \widehat{T}, \widehat{\Pi}$ and initialize as $\Pi_2 \leftarrow \Pi_1, T_2 \leftarrow T_1, \Phi_2 \leftarrow \Phi_1, \widehat{T} \leftarrow \emptyset, \widehat{\Pi} \leftarrow \emptyset$ and $T_{c_2} = T_{c_1}$.
{/* *$\widehat{\Pi}$ and $\widehat{T}$ denote the set of newly-added places and transitions to N_2; T_{c_2} (T_{u_2}) is the initial set of controllable (uncontrollable) transitions in T_2* */}
3: **for** $p \in \Pi_1$ **do**
4: **if** ($card(p^\bullet) > 1$) && (${}^\bullet(p^\bullet) \neq \{p\}$) **then**
5: **for** $t \in p^\bullet$ **do**
6: **if** ${}^\bullet t \neq \{p\}$ **then**
7: Introduce a new transition $\widehat{t}$ and a new place $\widehat{p}$. Let, $\widehat{T} \leftarrow \widehat{T} \cup \{\widehat{t}\}, \widehat{\Pi} \leftarrow \widehat{\Pi} \cup \{\widehat{p}\}, T_2 \leftarrow T_2 \cup \{\widehat{t}\}, \Pi_2 \leftarrow \Pi_2 \cup \{\widehat{p}\}$, and $\Phi_2 \leftarrow (\Phi_2 - \{(p,t)\}) \cup \{(p,\widehat{t}), (\widehat{t},\widehat{p}), (\widehat{p},t)\}$.
8: **end if**
9: **end for**
10: **end if**
11: **end for**
12: $T_{c_2} = T_{c_2} \cup \widehat{T}$
13: Return (N_2, T_{c_2}) where $N_2 = (\Pi_2, T_2, \Phi_2)$.
14: **else**
15: Exit { /* The procedure is inapplicable if $N_1 \notin \mathcal{G}$ */}
16: **end if**

Fig. 3. Procedure that converts a PN structure $N_1 = (\Pi_1, T_1, \Phi_1)$ ($N_1 \in \mathcal{G}$ [cf. (3)] into a PN structure $N_2 = (\Pi_2, T_2, \Phi_2)$ that is FC.

in Fig. 3 takes N_1 and T_{c_1} as input and returns a PN structure $N_2 = (\Pi_2, T_2, \Phi_2)$ that is FC by the addition of extra places $\widehat{\Pi}$ and transitions $\widehat{T}$, along with the set of controllable transitions $T_{c_2} (= T_{c_1} \cup \widehat{T})$. Consequently, $T_{u_2} = T_{u_1}$. From the construction, we have $\text{card}(\widehat{\Pi}) = \text{card}(\widehat{T})$, and $\forall \widehat{t} \in \widehat{T}, \text{card}(({}^\bullet\widehat{t})_{N_2}) = \text{card}((({}^\bullet\widehat{t})^\bullet_{N_2})_{N_2}) = 1$.

When the procedure in Fig. 3 is used on the non-FC structure N_1 shown in Fig. 1(a), we get the FC structure N_2 shown in Fig. 1(b). Specifically for this instance, $\widehat{\Pi} = \{p_{10}, p_{11}\}, \widehat{T} = \{t_{16}, t_{17}\}, T_{c_2} = \{t_3, t_4, t_{11}, t_{12}, t_{13}, t_{16}, t_{17}\}$, and $T_{u_2} = \{t_1, t_2, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{14}, t_{15}\} (= T_{u_1})$.

We now present a mapping between sets of markings of N_1 and N_2 that will find use in subsequent text. The function $\Theta : \mathcal{N}^{\text{card}(\Pi_1)} \rightarrow \mathcal{N}^{\text{card}(\Pi_2)}$ returns a marking $\mathbf{m}_2^i (= \Theta(\mathbf{m}_1^i))$ of N_2 when presented with a marking $\mathbf{m}_1^i$ of N_1, according to the following equation:

$$\mathbf{m}_2^i(p) \left(= \Theta(\mathbf{m}_1^i)(p)\right) = \begin{cases} \mathbf{m}_1^i(p) & \text{if } p \in \Pi_1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We will also find use for the function $f : T_1^* \rightarrow T_2^*$, where $f(\lambda) = \lambda$, for any $\sigma \in T_1^*, t \in T_1$, $f(\sigma t) = f(\sigma) f(t)$

$$f(t) = \begin{cases} t & \text{if } t \notin (\widehat{\Pi}^\bullet)_{N_2} \\ \omega_t t & \text{otherwise} \end{cases} \tag{5}$$

ω_t is some (fixed) $\text{card}(({}^\bullet t)_{N_2} \cap \widehat{\Pi})$-long string of transitions in the set $\{\widehat{t} \in \widehat{T} \mid \{t\} = ((\widehat{t}^\bullet)^\bullet_{N_2})_{N_2}\}$, and λ is the empty string. We suggest picking any one of the $(\text{card}(({}^\bullet t)_{N_2} \cap \widehat{\Pi}))!$-many candidates for ω_t for each $t \in (\widehat{T}^\bullet)_{N_2}$.

The following Lemma notes that if a PN $N_1 \in \mathcal{G}$, along with its set of controllable transitions, is presented as input to the procedure in Fig. 3, which results in an FC structure N_2, then in N_2, the set of uncontrollable transitions cannot share a common input place with any of the newly introduced transitions $\widehat{T}$. It also makes a specific note with regard to the nature of the function $f(\bullet)$ defined in (5), when the argument is an uncontrollable transition of N_1.

Lemma 3.1: Let $N_1 = (\Pi_1, T_1, \Phi_1)$ be an arbitrary PN structure in class $\mathcal{G}$, where the set of controllable transitions is $T_{c_1} \subseteq T_1$, and $N_2 = (\Pi_2, T_2, \Phi_2)$ is the FC structure that results when the procedure in Fig. 3 is used on N_1 and T_{c_1}, then

1) $\forall t_u \in T_{u_1} (= T_{u_2}), f(t_u) = t_u$, where $f(\bullet)$ is the function defined in (5).
2) $\forall t_u \in T_{u_2} (= T_{u_1}), ({}^\bullet t_u)_{N_2} \cap ({}^\bullet\widehat{T})_{N_2} = \emptyset$.

Proof: The construction procedure in Fig. 3 adds a transition $\widehat{t} \in \widehat{T}$ and a place $\widehat{p} \in \widehat{\Pi}$ to N_2 such that $\{(\widehat{t},\widehat{p}), (\widehat{p}, t_2)\} \subseteq \Phi_2$ for some $t_2 \in T_1$ if and only if one of the input places in $({}^\bullet t_2)_{N_1}$ violates the FC requirement [cf. (1)].

Since $N_1 \in \mathcal{G}$, we have $\forall t_u \in T_{u_1}, \forall t_1 \in T_1 - \{t_u\}$, $\exists p \in \Pi_1$, such that $(({}^\bullet t_1)_{N_1} \cap ({}^\bullet t_u)_{N_1} \neq \emptyset) \Rightarrow (({}^\bullet t_1)_{N_1} = ({}^\bullet t_u)_{N_1} = \{p\})$. That is, the input place of t_u does not violate the FC requirement. Therefore, $t_u \notin (\widehat{\Pi}^\bullet)_{N_2}$, and $f(t_u) = t_u$ [cf. (5)]. Additionally, since none of the transitions that are in conflict with t_u in N_1 has an input place that violates the FC requirement, it follows that $({}^\bullet t_u)_{N_2} \cap ({}^\bullet\widehat{T})_{N_2} = \emptyset$.

The Lemma listed in the succeeding discussion is about the existence of a valid firing string in N_2 that corresponds to each valid firing string σ_1 in N_1. This Lemma can be established by an induction argument over the length of the string σ_1 along with the specific details of the construction procedure in Fig. 3 and is skipped in the interest of space limitations.

Lemma 3.2: Let $N_1 = (\Pi_1, T_1, \Phi_1)$ be an arbitrary PN structure, where the set of controllable transitions is $T_{c_1} \subseteq T_1$, and $N_2 = (\Pi_2, T_2, \Phi_2)$ is the FC structure that results when the procedure in Fig. 3 is used on N_1 and T_{c_1}. Then, $\mathbf{m}_1^0 \rightarrow \sigma_1 \rightarrow \mathbf{m}_1^i$ in N_1 if and only if $\Theta(\mathbf{m}_1^0) \rightarrow f(\sigma_1) \rightarrow \Theta(\mathbf{m}_1^i)$ in N_2.

We now present one of the main results of this paper.

Theorem 3.3: Let $N_1 = (\Pi_1, T_1, \Phi_1)$ be an arbitrary PN structure in class $\mathcal{G}$, where the set of controllable (uncontrollable) transitions is denoted by $T_{c_1} \subseteq T_1$ ($T_{u_1} \subseteq T_1$). There is a supervisory policy that enforces liveness in $N_1(\mathbf{m}_1^0)$ if and only if there is a similar policy for $N_2(\Theta(\mathbf{m}_1^0))$, where $N_2 = (\Pi_2, T_2, \Phi_2)$ is the FC structure that results from the application of the procedure shown in Fig. 3 with N_1 and T_{c_1} presented as input.

Proof: (*Only if*) If there is a supervisory policy that enforces liveness in the PN $N_1(\mathbf{m}^0)$, there is a control-invariant set of markings $\mathcal{M}_1 \subseteq \Re(N_1(\mathbf{m}^0))$ that meets the requirement of Theorem 2.1. We will show that $\mathcal{M}_2 = \{\mathbf{m}_2^i \in \mathcal{N}^{\mathrm{card}(\Pi_2)} \mid \exists \mathbf{m}_1^i, \mathbf{m}_1^{i+1} \in \mathcal{M}_1$, where $\mathbf{m}_1^i \rightarrow t \rightarrow \mathbf{m}_1^{i+1}$ and $\Theta(\mathbf{m}_1^i) \rightarrow \sigma_2 \rightarrow \mathbf{m}_2^i$ for some $\sigma_2 \in \mathrm{pr}(f(t))\}$, where $\mathrm{pr}(\bullet)$ denotes the prefix set of the string argument, is control invariant with respect to N_2 and meets the additional requirements in Theorem 2.1, which implies the existence of a supervisory policy that enforces liveness in $N_2(\Theta(\mathbf{m}_1^0))$.

Suppose $\mathbf{m}_2^i \rightarrow t_u \rightarrow \mathbf{m}_2^j$ in N_2, where $\mathbf{m}_2^i \in \mathcal{M}_2$ and $t_u \in T_{u_2} (= T_{u_1})$. We now establish the control invariance of $\mathcal{M}_2$ by showing $\mathbf{m}_2^j \in \mathcal{M}_2$. Since $\mathbf{m}_2^i \in \mathcal{M}_2$, it follows that $\exists \mathbf{m}_1^i, \mathbf{m}_1^{i+1} \in \mathcal{M}_1, \exists t \in T_1$ such that $\mathbf{m}_1^i \rightarrow t \rightarrow \mathbf{m}_1^{i+1}$ in N_1 $(\rightarrow \Theta(\mathbf{m}_1^i) \rightarrow f(t) \rightarrow \Theta(\mathbf{m}_1^{i+1})$ (cf. Lemma 3.2), and $\Theta(\mathbf{m}_1^i) \rightarrow \sigma_2 \rightarrow \mathbf{m}_2^i$ in N_2, where $\sigma_2 \in \mathrm{pr}(f(t))$. We claim that $t_u \in T_e(N_2, \Theta(\mathbf{m}_1^i)) \cap T_e(N_2, \Theta(\mathbf{m}_1^{i+1}))$.

Since $N_1 \in \mathcal{G}$, the construction procedure in Fig. 3 ensures that $t_u \notin (\widehat{\Pi}^\bullet)_{N_2}$. Hence, if an input place to t_u is empty at the marking $\Theta(\mathbf{m}_1^i)$, then it should remain empty at the marking $\mathbf{m}_2^i$. Since $t_u \in T_e(N_2, \mathbf{m}_2^i)$, it follows that $t_u \in T_e(N_2, \Theta(\mathbf{m}_1^i))$.

Since $N_1 \in \mathcal{G}$, we have $({}^\bullet t_u)_{N_2} \cap ({}^\bullet \widehat{T})_{N_2} = \emptyset$ [cf. Lemma 3.1 (2)], and the construction procedure in Fig. 3 ensures that $({}^\bullet t_u)_{N_2} \cap \widehat{\Pi} = \emptyset$. Hence, if some place in $({}^\bullet t_u)_{N_2}$ is nonempty at some marking, it remains nonempty if any transition in $\widehat{T}$ or $(\widehat{\Pi}^\bullet)_{N_2}$ were to fire. Therefore, $t_u \in T_e(N_2, \Theta(\mathbf{m}_1^{i+1}))$. That is, the firing of t_u in N_2 does not influence the firing of the transitions that constitute $f(t)$, and vice-versa.

Hence, $\Theta(\mathbf{m}_1^i) \rightarrow t_u \rightarrow \Theta(\mathbf{m}_1^{i+2}) \rightarrow f(t) \rightarrow \Theta(\mathbf{m}_1^{i+3})$ and $\Theta(\mathbf{m}_1^i) \rightarrow f(t) \rightarrow \Theta(\mathbf{m}_1^{i+1}) \rightarrow t_u \rightarrow \Theta(\mathbf{m}_1^{i+3})$ in N_2. Additionally, $\Theta(\mathbf{m}_1^{i+2}) \rightarrow \sigma_2 \rightarrow \mathbf{m}_2^j$ in N_2. From Lemma 3.2, we have $\mathbf{m}_1^i \rightarrow t_u \rightarrow \mathbf{m}_1^{i+2} \rightarrow t \rightarrow \mathbf{m}_1^{i+3}$ and $\mathbf{m}_1^i \rightarrow t \rightarrow \mathbf{m}_1^{i+1} \rightarrow t_u \rightarrow \mathbf{m}_1^{i+3}$ in N_1. Since $\mathbf{m}_1^i, \mathbf{m}_1^{i+1} \in \mathcal{M}_1$ and $\mathcal{M}_1$ is control invariant, it follows that $\mathbf{m}_1^{i+2}, \mathbf{m}_1^{i+3} \in \mathcal{M}_1$, which, in turn, implies that $\mathbf{m}_2^j \in \mathcal{M}_2$, which establishes the control invariance of $\mathcal{M}_2$.

From Theorem 2.1, we have $\mathbf{m}_1^0 \in \mathcal{M}_1$ and $\forall \mathbf{m}_1^i \in \mathcal{M}_1, \exists \mathbf{m}_1^{i+1}, \mathbf{m}_1^{i+2} \in \mathcal{M}_1, \exists \sigma_1, \sigma_2 \in T_1^*$ such that $\mathbf{m}_1^i \rightarrow \sigma_1 \rightarrow \mathbf{m}_1^{i+1} \rightarrow \sigma_2 \rightarrow \mathbf{m}_1^{i+2}$ in N_1, 1) $\mathbf{m}_1^{i+2} \geq \mathbf{m}_1^{i+1}$, 2) $\mathbf{x}(\sigma_2) \geq \mathbf{1}$, and 3) $\forall \sigma_3 \in \mathrm{pr}(\sigma_1\sigma_2)$, $(\mathbf{m}_1^i \rightarrow \sigma_3 \rightarrow \mathbf{m}_1^{i+3}) \Rightarrow (\mathbf{m}_1^{i+3} \in \mathcal{M}_1)$. From Lemma 3.2, we know that $\Theta(\mathbf{m}_1^i) \rightarrow f(\sigma_1) \rightarrow \Theta(\mathbf{m}_1^{i+1}) \rightarrow f(\sigma_2) \rightarrow \Theta(\mathbf{m}_1^{i+2})$ in N_2.

We first note that $\Theta(\mathbf{m}_1^0) \in \mathcal{M}_2$. Additionally, since $(\mathbf{m}_1^{i+2} \geq \mathbf{m}_1^{i+1}) \Rightarrow (\Theta(\mathbf{m}_1^{i+1}) \geq \Theta(\mathbf{m}_1^{i+1}))$. From the construction procedure in Fig. 3 and the fact that $\mathbf{x}(\sigma_2) \geq \mathbf{1}$, we infer that every member of T_2 appears at least once in $f(\sigma_2)$. Specifically, if some $\widehat{t} \in \widehat{T}$ does not appear in $f(\sigma_2)$, then $\exists t \in T_1 (\subseteq T_2), \exists \widehat{p} \in \Pi_2$ such that $\widehat{p} \in^\bullet t$ and ${}^\bullet\widehat{p} = \{\widehat{t}\}$. This would mean that the transition t cannot appear in $f(\sigma_2)$, which is a contradiction.

Hence, $\mathcal{M}_2$ meets each requirement in Theorem 2.1, which, in turn, means there is a supervisory control policy that enforces liveness in $N_2(\Theta(\mathbf{m}_1^0))$.

(*If*) The existence of a supervisory policy that enforces liveness in $N_2(\Theta(\mathbf{m}_1^0))$ implies the existence of a set of markings $\mathcal{M}_2 \subseteq \Re(N_2(\Theta(\mathbf{m}_1^0)))$ that is control invariant with respect to N_2 and that also satisfies the requirements of Theorem 2.1. Let $\mathcal{M}_1 = \{\mathbf{m}_1^i \in \mathcal{N}^{\mathrm{card}(\Pi_1)} \mid \Theta(\mathbf{m}_1^i) \in \mathcal{M}_2\}$. We will show that $\mathcal{M}_1$ is control invariant with respect to N_1 and satisfies the requirements of Theorem 2.1.

Suppose $\mathbf{m}_1^i \rightarrow t_u \rightarrow \mathbf{m}_1^{i+1}$ in N_1, where $\mathbf{m}_1^i \in \mathcal{M}_1$ and $t_u \in T_{u_1} (= T_{u_2})$. From the definition of $\mathcal{M}_1$, we have $\Theta(\mathbf{m}_1^i) \in \mathcal{M}_2$. From Lemma 3.1(1) and Lemma 3.2, we know $\exists \sigma_2 \in T_2^*$ such that $\Theta(\mathbf{m}_1^i) \rightarrow t_u \rightarrow \Theta(\mathbf{m}_1^{i+1})$ in N_2. Since $\mathcal{M}_2$ is control invariant with respect to N_2, it follows that $\Theta(\mathbf{m}_1^{i+1}) \in \mathcal{M}_2$, which, in turn, implies that $\mathbf{m}_1^{i+1} \in \mathcal{M}_1$. Therefore, $\mathcal{M}_1$ is control invariant with respect to N_1.

From Theorem 2.1, we have $\Theta(\mathbf{m}_1^0) \in \mathcal{M}_2$ and $\forall \mathbf{m}_2^i \in \mathcal{M}_2, \exists \mathbf{m}_2^{i+1}, \mathbf{m}_2^{i+2} \in \mathcal{M}_2, \exists \sigma_1, \sigma_2 \in T_2^*$ such that $\mathbf{m}_2^i \rightarrow \sigma_1 \rightarrow \mathbf{m}_2^{i+1} \rightarrow \sigma_2 \rightarrow \mathbf{m}_2^{i+2}$ in N_2, 1) $\mathbf{m}_2^{i+2} \geq \mathbf{m}_2^{i+1}$, 2) $\mathbf{x}(\sigma_2) \geq \mathbf{1}$, and 3) $\forall \sigma_3 \in \mathrm{pr}(\sigma_1\sigma_2), (\mathbf{m}_2^i \rightarrow \sigma_3 \rightarrow \mathbf{m}_2^{i+3}) \Rightarrow (\mathbf{m}_2^{i+3} \in \mathcal{M}_2)$.

From the definition of $\mathcal{M}_1$, we have $\mathbf{m}_1^0 \in \mathcal{M}_1$. Additionally, $\forall \mathbf{m}_1^i \in \mathcal{M}_1$, we have $\Theta(\mathbf{m}_1^i) \in \mathcal{M}_2$, and from the observations made earlier, $\exists \mathbf{m}_2^{i+1}, \mathbf{m}_2^{i+2} \in \mathcal{M}_2$, $\exists \sigma_1, \sigma_2 \in T_2^*$, such that $\Theta(\mathbf{m}_1^i) \rightarrow \sigma_1 \rightarrow \mathbf{m}_2^{i+1} \rightarrow \sigma_2 \rightarrow \mathbf{m}_2^{i+2}$ in N_2, 1) $\mathbf{m}_2^{i+2} \geq \mathbf{m}_2^{i+1}$, 2) $\mathbf{x}(\sigma_2) \geq \mathbf{1}$, and 3) $\forall \sigma_3 \in \mathrm{pr}(\sigma_1\sigma_2), (\mathbf{m}_2^i \rightarrow \sigma_3 \rightarrow \mathbf{m}_2^{i+3}) \Rightarrow (\mathbf{m}_2^{i+3} \in \mathcal{M}_2)$. Let $\widetilde{T}_2 = \{t \in T_2 \mid \mathrm{card}({}^\bullet t)_{N_1} > 1$ and $({}^\bullet t)^\bullet_{N_1} \neq \{t\}\}$. From the procedure in Fig. 3, we have $\forall t \in \widetilde{T}_2, \exists p \in \Pi_2 - \widehat{\Pi} (= \Pi_1), \exists \widehat{p} \in \widehat{\Pi}, \exists \widehat{t} \in \widehat{T}$ such that $\{(p, \widehat{t}), (\widehat{t}, \widehat{p}), (\widehat{p}, t)\} \subset \Phi_2$. Since all elements of $\widetilde{T}_2$ appear at least once in σ_2, without loss of generality, we can suppose that $\forall \widehat{p} \in \widehat{\Pi}, \mathbf{m}_2^{i+1}(\widehat{p}) = \mathbf{m}_2^{i+2}(\widehat{p}) = 0$. Therefore, $\mathbf{m}_2^{i+1} = \Theta(\mathbf{m}_1^{i+1})$ and $\mathbf{m}_2^{i+2} = \Theta(\mathbf{m}_1^{i+1})$ for some $\mathbf{m}_1^{i+1}$ and $\mathbf{m}_1^{i+2}$, respectively. In turn, this implies $\mathbf{m}_1^{i+1}, \mathbf{m}_1^{i+2} \in \mathcal{M}_1$ can be shown to meet the remaining requirements of Theorem 2.1, which, in turn, implies that there is a supervisory policy that enforces liveness in $N_1(\mathbf{m}_1^0)$.

The following observation is a direct consequence of (1) and (3) and the fact that N_1 in Fig. 1(a), which is not an FC structure, belongs to the family $\mathcal{G}$. Additionally, every FC structure belongs to $\mathcal{G}$ as the implicant of (3) is never true for an FC structure.

Corollary 3.4: The family of partially controlled PNs $\mathcal{G}$ defined earlier strictly includes the set of partially controlled FC structures.

The existence of a supervisory policy that enforces liveness in an arbitrary FCPN is decidable [1]. This, along with Theorem 3.3, implies that the existence of a supervisory policy that enforces liveness in members of $\mathcal{G}$ is decidable. This is formally stated in the succeeding discussion as one of the main results of this paper.

Theorem 3.5: The existence of a supervisory policy that enforces liveness for any PN $N(\mathbf{m}^0)$, where $N \in \mathcal{G}$, is decidable.

Since testing the existence of a supervisory policy in an arbitrary FCPN is *nondeterministic polynomial time hard* [1], the same is true for the larger class of PNs whose structure belongs to $\mathcal{G}$. It is known that, when there is a supervisory policy that enforces liveness in an FCPN $N(\mathbf{m}^0)$, then $\mathbf{m}^0 \in \Delta(N)$ [cf. (2)], and the minimally restrictive supervisory policy that enforces liveness essentially ensures that the set of markings reachable under its supervision remains within the set $\Delta(N)$ [1].

As an illustration, for the FC structure N_2 shown in Fig. 1(b), the control-invariant right-closed set $\Delta(N_2)$ is identified by the minimal elements $\min(\Delta(N_2))$

$$\begin{aligned}\{&(1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0)^T, (1\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0)^T,\\ &(1\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0)^T, (0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0)^T,\\ &(0\,1\,0\,0\,0\,0\,1\,0\,0\,0\,0)^T, (0\,0\,1\,0\,0\,1\,0\,0\,0\,0\,0)^T,\\ &(0\,0\,1\,0\,0\,0\,0\,1\,0\,0\,0)^T, (0\,0\,0\,0\,0\,1\,1\,0\,0\,0\,0)^T,\\ &(0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,0)^T\}. \end{aligned} \tag{6}$$

Consequently, the (marking monotone) supervisory policy that prevents the firing of a controllable transition, whenever its firing would result in a marking that is not in $\Delta(N_2)$, is the minimally restrictive supervisory policy that enforces liveness in $N_2(\mathbf{m}_2^0)$ for any $\mathbf{m}_2^0 \in \Delta(N_2)$. The control invariance of $\Delta(N_2)$ guarantees the firing of any uncontrollable transition from any marking in $\Delta(N_2)$ will result in a new marking that is also in $\Delta(N_2)$.

If we let $\mathcal{M}_2 = \Delta(N_2)$, then the set $\mathcal{M}_1 = \{\mathbf{m}_3^i \in \mathcal{N}^{\text{card}(\Pi_3)} \mid \Theta(\mathbf{m}_3^i) \in \mathcal{M}_2\}$, which is introduced in the "if" part of the proof of Theorem 3.3, is a right-closed set whose minimal elements were identified earlier in (7). Furthermore, $\Delta(N_1) = \mathcal{M}_1$, and the (marking monotone) supervisory policy that ensures the set of reachable markings under supervision remains in the set $\Delta(N_1)(= \mathcal{M}_1)$ enforces liveness in $N_1(\mathbf{m}_1^0)$, for any $\mathbf{m}_1^0 \in \Delta(N_1)(= \mathcal{M}_1)$. Furthermore, this is a minimally restrictive supervisory policy that enforces liveness in $N_1(\mathbf{m}_1^0)$.

For the PN structure $N_1 \in \mathcal{G}$ shown in Fig. 1(a), the set $\Delta(N_1)$ is identified by the minimal elements $\min(\Delta(N_1))$

$$\begin{aligned}\big\{&(1\,1\,0\,0\,0\,0\,0\,0\,0)^T, (1\,0\,0\,0\,0\,0\,1\,0\,0\,0)^T,\\ &(1\,0\,0\,0\,0\,0\,0\,1\,0)^T, (0\,1\,1\,0\,0\,0\,0\,0\,0)^T,\\ &(0\,1\,0\,0\,0\,0\,1\,0\,0)^T, (0\,0\,1\,0\,0\,1\,0\,0\,0)^T,\\ &(0\,0\,1\,0\,0\,0\,0\,1\,0)^T, (0\,0\,0\,0\,0\,1\,1\,0\,0)^T,\\ &(0\,0\,0\,0\,0\,0\,1\,1\,0)^T\big\}.\end{aligned} \tag{7}$$

The minimally restrictive supervisory policy that enforces liveness in $N_1(\mathbf{m}_1^0)$, where $\mathbf{m}_1^0 \in \Delta(N_1)$, is the marking monotone policy $\mathcal{P} : \mathcal{N}^9 \times T \to \{0, 1\}$ defined by

$$\mathcal{P}(\mathbf{m}^i, t) = \begin{cases} 0 & \text{if } \mathbf{m}^i \in \Delta(N_1)\mathbf{m}^i \to t \to \mathbf{m}^{i+1} \text{ in } N \\ & \text{and } \mathbf{m}^{i+1} \notin \Delta(N_1) \\ 1 & \text{otherwise.} \end{cases}$$

Note that the control invariance of $\Delta(N_1)$ guarantees the fact that $(\mathcal{P}(\mathbf{m}^i, t) = 0) \to t \in \{t_4, t_{11}, t_{13}\}(\subseteq T_c)$.

Corollary 3.6: When it exists, the minimally restrictive supervisory policy that enforces liveness in any member of the family of partially controlled PNs $\mathcal{G}$ is a marking monotone policy that ensures the set of reachable markings under supervision belong to a right-closed set of markings.

The tokens in a concurrent system that is modeled by PNs typically represent resources that are to be consumed by tasks, which are identified by transitions. A *token reservation policy* is one where a token in a place is assigned to an unique output transition. From its structure, it follows that, if there is a supervisory policy that enforces liveness in an FCPN, then there is a token reservation policy that does the same. As a consequence of Theorem 3.3, we note that, if there is a liveness enforcing policy in a PN $N_1(\mathbf{m}_1^0)$, where $N_1 \in \mathcal{G}$, then there is a liveness enforcing token reservation policy for $N_1(\mathbf{m}_1^0)$. Specifically, each firing of a newly added transition in $\widehat{T}$ of $N_2(\mathbf{m}_2^0)$ corresponds to a token reservation operation.[3]

We close this section with an example that highlights the attributes of the class $\mathcal{G}$ by using the PN $N_3(\mathbf{m}_3^0)$ shown in Fig. 2. As noted earlier, $N_3 \notin \mathcal{G}$. If we were to use the procedure shown in Fig. 3 on N_3 and $T_{3_c} = \{t_2\}$, with the membership test in line 1 disabled, we would get the FC structure N_4 shown in Fig. 2(b). The supervisory policy that control enables all transitions in $N_3(\mathbf{m}_3^0)$ enforces liveness. However, there is no supervisory policy that enforces liveness in the FCPN $N_4(\mathbf{m}_4^0)$.

[3]We thank an anonymous reviewer of this paper for bringing this to our attention.

IV. Conclusion

A *PN* structure is *FC* if every arc from a place to a transition is either the unique output arc from the place or is the unique input arc to the transition. We have identified a family of partially controlled PNs, which is denoted by $\mathcal{G}$, that strictly contains the set of partially controlled FCPNs, for which the existence of a supervisory policy that enforces liveness is decidable. We have shown that, when there is a supervisory policy that enforces liveness in an instance of $\mathcal{G}$, the minimally restrictive supervisory policy that enforces liveness is characterized by a right-closed set of markings. This result implies that the procedures for the computation of the minimally restrictive LESP presented in [13] and [14] can be used for any member of the class $\mathcal{G}$. We suggest investigations into other families of PNs with this right-closure property as a direction for future research.

References

[1] R. Sreenivas, "On supervisory policies that enforce liveness in partially controlled free-choice Petri nets," *IEEE Trans. Autom. Control*, vol. 57, no. 2, pp. 435–449, Feb. 2012.

[2] B. Alpern and F. B. Schneider, "Defining liveness," *Inf. Process. Lett.*, vol. 21, no. 4, pp. 181–185, Oct. 1985.

[3] J. Peterson, *Petri Net Theory and the Modeling of Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1981.

[4] R. Sreenivas, "On the existence of supervisory policies that enforce liveness in discrete-event dynamic systems modeled by controlled Petri nets," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 928–945, Jul. 1997.

[5] T. Murata, "Petri nets: Properties, analysis and applications," *Proc. IEEE*, vol. 77, no. 4, pp. 541–580, Apr. 1989.

[6] A. Giua, "Petri nets as discrete event models for supervisory control," Ph.D. dissertation, ECSE Dept., Rensselaer Polytech. Inst., Troy, NY, 1992.

[7] J. Moody and P. Antsaklis, *Supervisory Control of Discrete Event Systems Using Petri Nets*. Norwell, MA: Kluwer, 1998.

[8] M. Iordache and P. Ansaklis, *Supervisory Control of Concurrent Systems: A Petri Net Structural Approach*. Norwell, MA: Kluwer, 2006.

[9] S. Reveliotis, *Real-Time Management of Resource Allocation Systems: A Discrete-Event Systems Approach*. New York: Springer-Verlag, 2005.

[10] S. Reveliotis, E. Roszkowska, and J. Choi, "Generalized algebraic deadlock avoidance policies for sequential resource allocation systems," *IEEE Trans. Autom. Control*, vol. 52, no. 12, pp. 2345–2350, Dec. 2007.

[11] A. Ghaffari, N. Rezg, and X. Xie, "Design of a live and maximally permissive Petri net controller using the theory of regions," *IEEE Trans. Robot. Autom.*, vol. 19, no. 1, pp. 137–142, Jan. 2003.

[12] N. Somnath and R. Sreenivas, "On deciding the existence of a liveness enforcing supervisory policy in a class of partially-controlled general free-choice Petri nets," *IEEE Trans. Autom. Sci. Eng.*, to be published.

[13] S. Chandrasekaran and R. Sreenivas, "A software tool for the synthesis of supervisory policies that avoid livelocks in Petri net models of manufacturing- and service-systems," in *Proc. 16th Annu. Conf. SOM*, New Delhi, India, Dec. 2012.

[14] S. Chandrasekaran and R. Sreenivas, "On the automatic generation of the minimally restrictive liveness enforcing supervisory policy for manufacturing- and service-systems modeled by a class of general free choice Petri nets," in *Proc. IEEE ICNSC*, Paris, France, Apr. 2013.